DEFORMING NAMIKAWA'S FIBER PRODUCTS

MICHELE ROSSI

Abstract. The aim of the present paper is producing examples supporting the conclusion of Y. Namikawa in Remark 2.8 of [30] and improving considerations of Example 1.11 of the same paper in the context of deformations of small geometric transitions to conifold transitions.

Contents

Let X be a complex projective threefold with terminal singularities and admitting a *small* resolution $\widehat{X} \stackrel{\phi}{\longrightarrow} X$ such that $\widehat{X}$ is a *Calabi–Yau threefold* (in the sense of Definition 1.1), where "small" means that the exceptional locus $\mathrm{Exc}(\phi)$ has codimension greater than or equal to two. Then it is well known that $\mathrm{Exc}(\phi)$ consists of a finite disjoint union of trees of rational curves of A–D–E type [37], [21], [34], [27], [7]. In his paper [30], Remark 2.8, Y. Namikawa considered the following

Problem. *When does $\widehat{X}$ have a flat deformation such that each tree of rational curves splits up into mutually disjoint* $(-1,-1)$–curves*?*

Let us recall that a $(-1,-1)$*–curve* is a rational curve in X whose normal bundle is isomorphic to $\mathcal{O}_{\mathbb{P}^1}(-1) \oplus \mathcal{O}_{\mathbb{P}^1}(-1)$. It arises precisely as exceptional locus of the resolution of an ordinary double point (a *node*) i.e. an isolated hypersurface singularity whose tangent cone is a non–degenerate quadratic cone.

Namikawa's problem is of considerable interest in the context of H. Clemens type problems of cycle deformations (see e.g. [7], Corollary (4.11)). Moreover, it is of significant interest in the context of geometric transitions and therefore in the study

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of Calabi–Yau threefolds moduli space. Let us recall that a *geometric transition* (g.t.) between two Calabi–Yau threefolds is the process obtained by "composing" a birational contraction to a normal threefold with a complex smoothing (see Definition 1.5). If the normal intermediate threefold has only nodal singularities then the considered g.t. is called a *conifold transition*. The interest in g.t.s goes back to the ideas of H. Clemens [5] and M. Reid [38] which gave rise to the so called *Calabi–Yau Web Conjecture* (see also [11] for a revised and more recent version) stating that (more or less) all Calabi–Yau threefolds can be connected to each other by means of a chain of g.t.s, giving a sort of (unexpected) "connectedness" of the Calabi–Yau threefolds moduli space. There is also a considerable physical interest in g.t.s owing to the fact that they connect topologically distinct models of Calabi–Yau vacua: the physical version of the Calabi–Yau Web Conjecture is a sort of (in this case expected) "uniqueness" of a space–time model for supersymmetric string theories (see e.g. [4] and references therein).

In this context, Namikawa's problem can then be rephrased as follows

Problem (for small geometric transitions)**.** *When does a small g.t. have a "flat deformation" to a conifold transition?*

Since the geometry of a general g.t. can be very intricate, while the geometry of a conifold transition is relatively easy and well understood as a topological surgery [5], the mathematical interest of such a problem is evident.

On the other hand, conifold transitions were the first (and among the few) g.t.s to be physically understood as a massive black holes condensation to a massless ones by A. Strominger [46]. Answering the given problem would then give a significant improvement in the physical interpretation of (at least the small) g.t.s bridging topologically distinct Calabi–Yau vacua.

Unfortunately in [30], Remark 2.8, Namikawa observed that a flat deformation positively resolving the given problem "*does not hold in general*" and produced an example of a *cuspidal fiber self–product of an elliptic rational surface with sections* whose resolution admits exceptional trees, composed of couples of rational curves intersecting at one point, which should not deform to a disjoint union of $(-1,-1)$–curves. Nevertheless, Example 1.11 in [30], supporting such a conclusion, contains an oversight forcing the given deformation to be actually a meaningless *trivial deformation* of the given cuspidal fiber product.

The present paper wil overcome such an oversight and produce *all the deformations* (which actually exist) of the Namikawa cuspidal fiber product supporting his conclusion in [30], Remark 2.8.

The paper is organized as follows.

In the first section we introduce notation, preliminaries and main facts needed throughout the paper and culminating with Proposition 1.12 which expresses the global change in topology induced by a small g.t.

The second section is dedicated to reviewing elliptic rational surfaces and their fiber products. The treatment will be as self–contained as possible, with some explicit examples. For more details the reader is referred to original papers by A. Kas, R. Miranda, U. Persson, C. Schoen and Y. Namikawa [17], [24], [25], [26], [44], [28]. See also the recent [16].

Section 3 is devoted to presenting the Namikawa construction of a fiber self–product of a particular elliptic rational surface with sections and singular "cuspidal" fibers (which will be called *cuspidal fiber product*). These are threefolds admitting six isolated singularities of Kodaira type $II \times II$ which have been very rarely studied in either the pioneering work of C. Schoen [44] or the recent [16]. For this reason, their properties, small resolutions and local deformations are studied in detail. In particular, all the local deformations induced by global versal deformations are studied in Proposition 3.3, while all the local deformations of a cuspidal singularity to three distinct nodes are studied in Proposition 3.5. They actually do not lift globally to the given small resolution, as stated by Theorem 3.6, revising the Namikawa considerations of [30], Remark 2.8 and Example 1.11. Finally, analytical and topological invariants of these fiber products and their resolutions and deformations are studied in Theorem 3.8 and reported in Table (60). The proof is based on the topology–changing properties of geometric transitions. This allows us to understand from a different perspective some known facts about fiber products of elliptic rational surfaces with sections (see e.g. Remarks 3.9 and 3.10).

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1. Preliminaries and notation

Let us start by recalling that there are a lot of more or less equivalent definitions of Calabi–Yau 3–folds e.g.: a Kähler complex, compact 3–fold admitting either (1) a *Ricci flat metric* (Calabi conjecture and Yau Theorem), or (2) a flat, non–degenerate, holomorphic 3–form, or (3) holonomy group a subgroup of SU(3) (see [15] for a complete description of equivalences and implications). In the present paper we will choose the following equivalent version of (3), which is, in the algebraic context, the analogue of smooth elliptic curves and smooth $K3$ surfaces.

Definition 1.1 (Calabi–Yau 3–folds)**.** A smooth, complex, projective 3–fold X is called *Calabi–Yau* if

(1) $\mathcal{K}_X \cong \mathcal{O}_X$,
(2) $h^{1,0}(X) = h^{2,0}(X) = 0$.

The standard example is the smooth quintic threefold in $\mathbb{P}^4$.

1.1. **The Picard number of a Calabi–Yau threefold.** Let X be a Calabi–Yau threefold and consider the *Picard group*

$$\mathrm{Pic}(X) := \langle \text{Invertible Sheaves on } X \rangle_{\mathbb{Z}} / \text{isomorphism} \cong H^1(X, \mathcal{O}_X^*) \ .$$

The Calabi–Yau conditions $h^1(\mathcal{O}_X) = h^2(\mathcal{O}_X) = 0$, applied to the cohomology of the *exponential sequence*, give then the canonical isomorphism

$$\mathrm{Pic}(X) \cong H^2(X, \mathbb{Z}) \ .$$

Therefore the *Picard number of* X, which is defined as $\rho(X) := \mathrm{rk}\,\mathrm{Pic}(X)$, turns out to be

$$\rho(X) = b_2(X) = b_4(X) = h^{1,1}(X)\ . \tag{1}$$

Since the interior $\mathcal{K}(X)$ of the *closed Kähler cone* $\overline{\mathcal{K}}(X)$ of X, generated in the Kleiman space $H^2(X,\mathbb{R})$ by the classes of nef divisors, turns out to be the cone generated by the Kähler classes, the Picard number $\rho(X) = h^{1,1}(X)$ turns out to be the *dimension of the Kähler moduli space of* X.

1.2. **Deformations of Calabi–Yau threefolds.** Let $\mathcal{X} \xrightarrow{f} B$ be a *flat*, surjective map of complex spaces such that B is connected and there exists a special point $o \in B$ whose fibre $X = f^{-1}(o)$ may be singular. Then $\mathcal{X}$ is called *a deformation family of* X. If the fibre $X_b = f^{-1}(b)$ is smooth, for some $b \in B$, then X_b is called *a smoothing of* X.
Let Ω_X be the sheaf of holomorphic differential forms on X and consider the *Lichtenbaum–Schlessinger cotangent sheaves* [22] of X, $\Theta^i_X = \mathcal{E}xt^i(\Omega_X,\mathcal{O}_X)$. Then $\Theta^0_X = \mathcal{H}om\,(\Omega_X,\mathcal{O}_X) =: \Theta_X$ is the "tangent" sheaf of X and Θ^i_X is supported over $\mathrm{Sing}(X)$, for any $i > 0$. Consider the associated *local* and *global deformation objects*

$$T^i_X := H^0(X,\Theta^i_X) \quad , \quad \mathbb{T}^i_X := \mathrm{Ext}^i\left(\Omega^1_X,\mathcal{O}_X\right)\ ,\ i = 0,1,2.$$

Then by the *local to global spectral sequence* relating the global Ext and sheaf $\mathcal{E}xt$ (see [12] and [8] II, 7.3.3) we get

$$E^{p,q}_2 = H^p\left(X,\Theta^q_X\right) \Longrightarrow \mathbb{T}^{p+q}_X$$

giving that

$$\mathbb{T}^0_X \cong T^0_X \cong H^0(X,\Theta_X)\ , \tag{2}$$

$$\text{if } X \text{ is smooth then} \quad \mathbb{T}^i_X \cong H^i(X,\Theta_X)\ , \tag{3}$$

$$\text{if } X \text{ is Stein then} \quad T^i_X \cong \mathbb{T}^i_X\ . \tag{4}$$

Given a deformation family $\mathcal{X} \xrightarrow{f} B$ of X *for each point* $b \in B$ *there is a well defined linear (and functorial) map*

$$D_b f:\ T_b B \longrightarrow \mathbb{T}^1_{X_b} \quad \text{(Generalized Kodaira–Spencer map)}$$

(see e.g. [33] Theorem 5.1). Recall that $\mathcal{X} \xrightarrow{f} B$ is called

- a *versal* deformation family of X if for any deformation family $(\mathcal{Y},X) \xrightarrow{g} (C,0)$ of X there exists a map of pointed complex spaces $h:(U,0)\to(B,o)$, defined on a neighborhood $0 \in U \subset C$, such that $\mathcal{Y}|_U$ is the *pull–back* of $\mathcal{X}$ by h i.e.

$$\begin{tikzcd} & \mathcal{Y}|_U = U\times_h\mathcal{X} \arrow[r] \arrow[d, "g"] & \mathcal{X} \arrow[d, "f"] \\ C & U \arrow[l, hook'] \arrow[r, "h"] & B \end{tikzcd}$$

- an *effective versal* deformation family of X if it is versal and the generalized Kodaira–Spencer map evaluated at $o \in B$, $D_o f:\ T_o B \longrightarrow \mathbb{T}^1_X$ is injective,

- a *universal* family if it is versal and h is univocally defined in a neighborhood $0 \in U \subset C$.

Theorem 1.2 (Douady–Grauert–Palamodov [6], [10], [32] and [33] Theorems 5.4 and 5.6)**.** *Every compact complex space X has an effective versal deformation $\mathcal{X} \xrightarrow{f} B$ which is a proper map and a versal deformation of each of its fibers. Moreover the germ of analytic space (B, o) is isomorphic to the germ of analytic space $(q^{-1}(0), 0)$, where $q : \mathbb{T}^1_X \to \mathbb{T}^2_X$ is a suitable holomorphic map (the* obstruction map*) such that $q(0) = 0$.*

In particular if $q \equiv 0$ (e.g. when $\mathbb{T}^2_X = 0$) then (B, o) turns out to be isomorphic to the germ of a neighborhood of the origin in $\mathbb{T}^1_X$.

Definition 1.3 (Kuranishi space and number)**.** The germ of analytic space

$$\mathrm{Def}(X) := (B, o)$$

as defined in Theorem 1.2, is called the *Kuranishi space of X*. The *Kuranishi number* $\mathrm{def}(X)$ *of X* is then the maximum dimension of irreducible components of $\mathrm{Def}(X)$.
$\mathrm{Def}(X)$ is said to be *unobstructed* or *smooth* if the obstruction map q is the constant map $q \equiv 0$. In this case $\mathrm{def}(X) = \dim_{\mathbb{C}} \mathbb{T}^1_X$.

Theorem 1.4 ([33] Theorem 5.5)**.** *If $\mathbb{T}^0_X = 0$ then the versal effective deformation of X, given by Theorem 1.2, is actually universal for all the fibres close enough to X.*

Let us now consider the case of a *Calabi–Yau threefold* X. By the Bogomolov–Tian–Todorov–Ran Theorem [3],[48],[49],[35] the Kuranishi space $\mathrm{Def}(X)$ *is smooth* and (3) gives that

$$\mathrm{def}(X) = \dim_{\mathbb{C}} \mathbb{T}^1_X = h^1(X, \Theta_X) = h^{2,1}(X) \tag{5}$$

where the last equality on the right is obtained by the Calabi–Yau condition $\mathcal{K}_X \cong \mathcal{O}_X$. Applying the Calabi–Yau condition once again gives $h^0(\Theta_X) = h^{2,0}(X) = 0$. Therefore (2) and Theorem 1.4 give the existence of a *universal effective family of Calabi–Yau deformations of X*. In particular $h^{2,1}(X)$ *turns out to be the dimension of the complex moduli space of X*.

1.3. **Geometric transitions.**

Definition 1.5 (see [39] and references therein)**.** Let $\widehat{X}$ be a Calabi–Yau threefold and $\phi : \widehat{X} \longrightarrow X$ be a *birational contraction* onto a *normal* variety. Assume that there exists a *Calabi–Yau smoothing* $\widetilde{X}$ of X. Then the process of going from $\widehat{X}$ to $\widetilde{X}$ is called a *geometric transition* (for short *transition* or *g.t.*) and denoted by $T(\widehat{X}, X, \widetilde{X})$ or by the diagram

$$\begin{tikzcd} \widehat{X} \arrow[r, "\phi"] \arrow[rr, dotted, bend right, "T"'] & X \arrow[r, leftrightsquigarrow] & \widetilde{X} \end{tikzcd} \tag{6}$$

The most well known example of a g.t. is given by a generic quintic threefold $X \subset \mathbb{P}^4$ containing a plane. One can check that $\mathrm{Sing}(X)$ is composed by 16 *ordinary double points*. Looking at the strict transform of X, in the blow–up of $\mathbb{P}^4$ along the contained plain, gives the resolution $\widehat{X}$, while a generic quintic threefold in $\mathbb{P}^4$ gives

the smoothing $\widetilde{X}$. Due to the particular nature of $\mathrm{Sing}(X)$ the g.t. $T(\widehat{X},X,\widetilde{X})$ is actually an example of a *conifold transition*.

Definition 1.6 (Conifold transitions)**.** A g.t. $T(\widehat{X},X,\widetilde{X})$ is called *conifold* if X admits only *ordinary double points* (nodes or o.d.p.'s) as singularities.

Definition 1.7 (Small geometric transitions)**.** A geometric transition

$$\begin{tikzcd} \widehat{X} \arrow[r, "\phi"] \arrow[rr, bend right, dotted, "T"'] & X \arrow[r, leftrightsquigarrow, dashed] & \widetilde{X} \end{tikzcd}$$

will be called *small* if $\mathrm{codim}_{\widehat{X}}\,\mathrm{Exc}(\phi) > 1$, where $\mathrm{Exc}(\phi)$ denotes the exceptional locus of ϕ.

As an obvious example, any conifold transition is a small g.t..

Remark 1.8. Let $T(\widehat{X},X,\widetilde{X})$ be a small g.t.. Then $\mathrm{Sing}(X)$ is composed at most by *terminal singularities of index* 1 which turns out to be *isolated hypersurface singularities* (actually of *compound Du Val type* [36], [37]). In [29], Theorem A, Y. Namikawa proved an extension of the Bogomolov–Tian–Todorov–Ran Theorem allowing to conclude that $\mathrm{Def}(X)$ *is smooth* also in the present situation. Therefore

$$\mathrm{def}(X) = \dim_{\mathbb{C}} \mathbb{T}^1_X \ . \tag{7}$$

Moreover the Leray spectral sequence of the sheaf $\phi_*\Theta_{\widehat{X}}$ gives

$$h^0(\Theta_X) = h^0(\phi_*\Theta_{\widehat{X}}) = h^0(\Theta_{\widehat{X}}) = h^{2,0}(\widehat{X}) = 0$$

where the first equality on the left is a consequence of the isomorphism $\Theta_X \cong \phi_*\Theta_{\widehat{X}}$ (see [7] Lemma (3.1)) and the last equality on the right is due to the Calabi–Yau condition for $\widehat{X}$. Then Theorem 1.4 and (2) allow to conclude that X *admits a universal effective family of Calabi–Yau deformations*.

Proposition 1.9 (Clemens' formulas [5])**.** *Let* $T(\widehat{X},X,\widetilde{X})$ *be a* conifold *transition and say* b_i *the* i*–th* Betti number *i.e. the rank of the* i*–th singular homology group. Then there exist two non negative integers* k, c *such that:*

(1) $N := |\,\mathrm{Sing}(X)| = k + c$;
(2) *(Betti numbers)* $b_i(\widehat{X}) = b_i(X) = b_i(\widetilde{X})$ *for* $i \neq 2,3,4$, *and*

$$\begin{array}{ccccc} b_2(\widehat{X}) & = & b_2(X)+k & = & b_2(\widetilde{X})+k \\ \| & & & & \| \\ b_4(\widehat{X}) & = & b_4(X) & = & b_4(\widetilde{X})+k \\ & & & & \\ b_3(\widehat{X}) & = & b_3(X)-c & = & b_3(\widetilde{X})-2c \end{array}$$

where vertical equalities are given by Poincaré Duality;
(3) *(Hodge numbers)*

$$\begin{aligned} h^{2,1}(\widetilde{X}) &= h^{2,1}(\widehat{X})+c \\ h^{1,1}(\widetilde{X}) &= h^{1,1}(\widehat{X})-k \end{aligned}$$

(4) *(Euler–Poincaré characteristic)*

$$\chi(\widehat{X}) = \chi(X)+N = \chi(\widetilde{X})+2N \ .$$

(for a detailed proof see [39] Theorem 3.3 and references therein).

Remark 1.10 (Meaning of k and c). The integers k and c admit topological, geometrical and physical interpretations. In *topology* k turns out to be the dimension of the subspace $\langle[\phi^{-1}(p)] \mid p \in \mathrm{Sing}(X)\rangle_{\mathbb{Q}}$ of $H_2(\widehat{X},\mathbb{Q})$, generated by the 2–cycles composing the exceptional locus of the birational resolution $\phi : \widehat{X} \longrightarrow X$. On the other hand c is the dimension of the subspace generated in $H_3(\widetilde{X},\mathbb{Q})$ by the vanishing cycles. In *geometry* equations (3) in the statement of Proposition 1.9 means that *a conifold transition decreases Kähler moduli by k and increases complex moduli by c*: this is a consequence of (1) and (5). At last (but not least) a conifold transition has been physically understood by A. Strominger [46] as the process connecting two topologically distinct Calabi–Yau vacua by means of a condensation of massive black holes to massless ones inducing a decreasing of k vector multiplets and an increasing of c hypermultiplets.

1.4. **Milnor and Tyurina numbers of isolated hypersurface singularities.** Let $\mathcal{O}_0$ be the local ring of germs of holomorphic function of $\mathbb{C}^{n+1}$ at the origin, which is the localization of the polynomial ring $\mathcal{O} := \mathbb{C}[x_1,\ldots,x_{n+1}]$ at the maximal ideal $\mathfrak{m}_0 := (x_1,\ldots,x_{n+1})$. By definition of holomorphic function and the identity principle we have that $\mathcal{O}_0$ *is isomorphic to the ring of convergent power series* $\mathbb{C}\{x_1,\ldots,x_{n+1}\}$. A *germ of hypersurface singularity* is defined as the Stein complex space

$$U_0 := \mathrm{Spec}(\mathcal{O}_{F,0})$$

where $\mathcal{O}_{F,0} := \mathcal{O}_0/(F)$ and F is the germ represented by a polynomial function.

Definition 1.11. The *Milnor number* of the hypersurface singularity $0 \in U_0$ is defined as *the multiplicity of* 0 *as solution of the system of partials of* F ([23] §7) which is

$$\mu(0) := \dim_{\mathbb{C}}(\mathcal{O}_0/J_F) = \dim_{\mathbb{C}}(\mathbb{C}\{x_1,\ldots,x_{n+1}\}/J_F)$$

as a $\mathbb{C}$–vector space, where J_F is the jacobian ideal $\left(\frac{\partial F}{\partial x_1},\ldots,\frac{\partial F}{\partial x_{n+1}}\right)$.
The *Tyurina number* of the hypersurface singularity $0 \in U_0$ is defined as *the Kuranishi number* $\mathrm{def}(U_0)$. Since U_0 is Stein, $\mathrm{Def}(U_0)$ is smooth and, by (4),

$$\begin{aligned}\tau(0) &:= \mathrm{def}(U_0) = \dim_{\mathbb{C}} \mathbb{T}^1_{U_0} = \dim_{\mathbb{C}} T^1_{U_0} = h^0(U_0,\Theta^1_{U_0}) \\ &= \dim_{\mathbb{C}}(\mathcal{O}_{F,0}/J_F) = \dim_{\mathbb{C}}(\mathbb{C}\{x_1,\ldots,x_{n+1}\}/(F)+J_F)\ .\end{aligned}$$

Then $\tau(0) \leq \mu(0)$. According with [43], $\tau(0) = \mu(0)$ *if and only if F is the germ of a weighted homogeneous polynomial.*

Proposition 1.12 (Generalized Clemens' formulas [31] Remark (3.8), [40] Theorem 7)**.** *Given a* small *g.t.* $T(\widehat{X},X,\widetilde{X})$ *there exist three non–negative integers* k,c',c'' *such that*

(1) *the total number of irreducible components of the exceptional locus* $E = \mathrm{Exc}(\phi)$ *is*
$$n := \sum_{p\in\mathrm{Sing}(X)} n_p = k + c'\ ,$$
(2) *the* global Milnor number of X *is*
$$m := \sum_{p\in\mathrm{Sing}(X)} \mu(p) = k + c''\ ,$$

(3) *(Betti numbers)* $b_i(\widehat{X}) = b_i(X) = b_i(\widetilde{X})$ *for* $i \neq 2, 3, 4$, *and*

$$\begin{array}{ccccc} b_2(\widehat{X}) & = & b_2(X) + k & = & b_2(\widetilde{X}) + k \\ \| & & & & \| \\ b_4(\widehat{X}) & = & b_4(X) & = & b_4(\widetilde{X}) + k \\ & & & & \\ b_3(\widehat{X}) & = & b_3(X) - c' & = & b_3(\widetilde{X}) - (c' + c'') \end{array}$$

where vertical equalities are given by Poincaré Duality,

(4) *(Hodge numbers)*

$$\begin{aligned} h^{1,1}(\widetilde{X}) &= h^{1,1}(\widehat{X}) - k \\ h^{2,1}(\widetilde{X}) &= h^{2,1}(\widehat{X}) + c \end{aligned}$$

where $c := (c' + c'')/2$.

(5) *(Euler–Poincaré characteristic)*

$$\chi(\widehat{X}) = \chi(X) + n = \chi(\widetilde{X}) + n + m \ .$$

Remark 1.13. In particular, if $T(\widehat{X}, X, \widetilde{X})$ is a *conifold* transition, then $c' = c'' = c$ and $|\operatorname{Sing}(X)| = k + c$ giving the previous Proposition 1.9.

Remark 1.14 (Picard number of the central fibre). The integer k have the same topological and geometric interpretations given in Remark 1.10. Also the integer $c = (c' + c'')/2$ have the same geometric meaning of increasing of complex moduli induced by a small g.t. For a topological interpretation look at points (1) and (2) in Proposition 1.12 which explain the role of integers c' and c''. Precisely by (1) c' gives the *number of independent linear relations linking the* 2*–cycles of irreducible components of* $\operatorname{Exc}(\phi)$ *in* $H_2(\widehat{X}, \mathbb{Q})$. Since these 2–cycles are contracted by ϕ down to the isolated singularities of X, the previous relations generate c' new independent 3–cycles in $H_3(X, \mathbb{Q})$. On the other hand c'' turns out to have the same topological meaning of c in Remark 1.10 which is the dimension of the subspace generated in $H_3(\widetilde{X}, \mathbb{Q})$ by the vanishing 3–cycles. Then by (2) k turns out to be *the number of independent linear relations linking the vanishing* 3*–cycles in* $H_3(\widetilde{X}, \mathbb{Q})$. Degenerating $\widetilde{X}$ to X shrinks the vanishing 3–cycles to the isolated singular points of X, then

(a) *the degeneration from* $\widetilde{X}$ *to* X *gives rise to* k *new independent* 4*–cycles in* $H_4(X, \mathbb{Q})$.

In [31] Y. Namikawa and J. Steenbrink observe that

(b) $k = b_4(X) - b_2(X)$, *called the* defect *of* X, *turns out to give the rank of the quotient group* $\operatorname{Cl}(X)/\operatorname{CaCl}(X)$

of Weil divisors (mod. linear equivalence) with respect to the subgroup of Cartier divisors (mod. linear equivalence) (notation as in [13] II.6) meaning that *the* k *new independent* 4*–cycles in* $H_4(X, \mathbb{Q})$ *are homology classes of Weil divisors on* X *which are not Cartier divisors.* Since X is a reduced, irreducible and normal threefold, $\operatorname{CaCl}(X) \cong \operatorname{Pic}(X)$ (see [13] Propositions II.3.1 and II.6.15). Then statements (a) and (b) and equations (1) allow to conclude that

$$\rho(X) = \rho(\widetilde{X}) = h^{1,1}(\widetilde{X}) \ . \tag{8}$$

2. Fiber products of rational elliptic surfaces with sections

In the present section we will review some well known facts about rational elliptic surfaces with section and their fiber products, with some explicit example. For further details the reader is referred to [25], [26] and [44].

2.1. **Blow–up of elliptic pencils and fiber products.** Let Y and Y' be *rational elliptic surfaces with sections* i.e. rational surfaces admitting elliptic fibrations over $\mathbb{P}^1$

$$r\ :\ Y \longrightarrow \mathbb{P}^1 \quad , \quad r'\ :\ Y' \longrightarrow \mathbb{P}^1$$

with distinguished sections σ_0 and σ_0', respectively (notation as in [44] and [26]). Define

$$X := Y \times_{\mathbb{P}^1} Y' \ . \tag{9}$$

Write S (resp. S') for the images of the singular fibers of Y (resp. Y') in $\mathbb{P}^1$.

Proposition 2.1.

(1) *The fiber product X is smooth if and only if $S \cap S' = \emptyset$. In particular, if smooth, X is a Calabi–Yau threefold* ([44] §2).
(2) *Y (resp. Y') is the blow–up of $\mathbb{P}^2$ at the base locus of a rational map $\varrho : \mathbb{P}^2 \dashrightarrow \mathbb{P}^1$ (resp. $\varrho' : \mathbb{P}^2 \dashrightarrow \mathbb{P}^1$)* ([26] Prop. 6.1).
(3) *If $Y = Y'$ is* sufficiently general *and $r(= r')$ admits at most nodal fibers, then there always exists a small projective resolution $\widehat{X}$ of X* ([44] Lemma (3.1)).

About the proof. This Proposition is very well-known. Here we give a proof of (2) and (3) whose geometric set up will be useful in the following.
For (2) let $x = (x_0 : x_1 : x_2)$ denote homogeneous coordinates on $\mathbb{P}^2$. Then there exist homogeneous cubic polynomials $a(x), b(x)$ (resp. $a'(x), b'(x)$) without common factors such that Y (resp. Y') turns out to be $\mathbb{P}^2$ blown up at the base locus $a(x) = b(x) = 0$ (resp. $a'(x) = b'(x) = 0$) of the rational map

$$\varrho(x) := (a(x) : b(x)) \in \mathbb{P}^1 \quad (\text{resp. } \varrho'(x) := (a'(x) : b'(x)) \in \mathbb{P}^1) \ .$$

(this is a well known fact: for more details the reader is referred to [24] §6, [26] and references therein). Explicitly Y (resp. Y') can be described as the following hypersurface

$$\mathbb{P}^2[x] \times \mathbb{P}^1[\lambda] \supset Y \ :\ \lambda_1 a(x) - \lambda_0 b(x) = 0 \quad (\text{resp. } Y' : \lambda_1 a'(x) - \lambda_0 b'(x) = 0) \tag{10}$$

which is obviously fibred over $\mathbb{P}^1$: the choice of an order for the finite number of blow–ups of $\mathbb{P}^2$, gives a distinguished section σ_0 as the exceptional divisor of the last blow–up. The following commutative diagram then holds

$$\begin{tikzcd}
 & & X \arrow[dl] \arrow[dr] & & \\
\mathbb{P}^2[x] \times \mathbb{P}^1[\lambda] & Y \arrow[l, hook'] \arrow[d, "\text{blow–up}"'] \arrow[dr, "r"] & & Y' \arrow[r, hook] \arrow[d, "\text{blow–up}"] \arrow[dl, "r'"'] & \mathbb{P}^2[x'] \times \mathbb{P}^1[\lambda] \\
 & \mathbb{P}^2[x] \arrow[r, dashed, "\varrho"] & \mathbb{P}^1[\lambda] & \mathbb{P}^2[x'] \arrow[l, dashed, "\varrho'"'] &
\end{tikzcd} \tag{11}$$

and X is birational to the bi–cubic hypersurface

$$\mathbb{P}^2[x] \times \mathbb{P}^2[x'] \supset W \ : \ a(x)b'(x') - a'(x')b(x) = 0 \ . \tag{12}$$

For a *sufficiently general* choice of $a, b, a', b' \in H^0(\mathcal{O}_{\mathbb{P}^2}(3))$, the singular locus $\mathrm{Sing}(W)$ is described by $a(x) = b(x) = a'(x') = b'(x') = 0$ and is composed by 81 o.d.p.'s. Then W admits a simultaneous small resolution $\widehat{W}$ obtained as the strict transform of W in the blow up of $\mathbb{P}^2 \times \mathbb{P}^2$ along the surface $a(x) = b(x) = 0$ (actually 9 copies of $\mathbb{P}^2$) and explicitly described by

$$\mathbb{P}^2[x] \times \mathbb{P}^2[x'] \times \mathbb{P}^1[\lambda] \supset \widehat{W} \ : \ \left\{ \begin{array}{c} \lambda_1 a(x) - \lambda_0 b(x) = 0 \\ \lambda_1 a'(x') - \lambda_0 b'(x') = 0 \end{array} \right. . \tag{13}$$

Then by (9), (10) and (13) the birational equivalence between X and W extends to give an isomorphism $X \cong \widehat{W}$.
Let us now assume that $Y = Y'$. Then $a = a'$ and $b = b'$ implying that equations (13) of $\widehat{W} \cong X$ can be rewritten as follows

$$\mathbb{P}^2[x] \times \mathbb{P}^2[x'] \times \mathbb{P}^1[\lambda] \supset \widehat{W} \ : \ \left\{ \begin{array}{c} \lambda_1 a(x) - \lambda_0 b(x) = 0 \\ \lambda_1 a(x') - \lambda_0 b(x') = 0 \end{array} \right. . \tag{14}$$

Consider the diagonal locus $\Delta := \{(x, x', \lambda) \in \mathbb{P}^2[x] \times \mathbb{P}^2[x'] \times \mathbb{P}^1[\lambda] \mid x = x'\} \cong \mathbb{P}^2 \times \mathbb{P}^1$ and let $\widehat{\mathbb{P}}$ be the blow–up of $\mathbb{P}^2 \times \mathbb{P}^2 \times \mathbb{P}^1$ along Δ. The associated exceptional locus E is then a $\mathbb{P}^1$–bundle over Δ. Notice that $\Delta \cap \widehat{W}$ is isomorphic to the diagonal Weil divisor of the fiber product $X = Y \times_{\mathbb{P}^1} Y$ which contains $\mathrm{Sing}(X)$. Let $\widehat{X}$ be the strict transform of $\widehat{W}$ in the blow–up $\widehat{\mathbb{P}}$. Since $\mathrm{Sing}(X)$ is entirely composed by nodes, $\widehat{X}$ is smooth and $\widehat{X} \longrightarrow \widehat{W} \cong X$ turns out to be a small resolution, proving (3). □

Remark 2.2. Observe that if $X = Y \times_{\mathbb{P}^1} Y'$ is smooth then $X \cong \widehat{W} \longrightarrow W$ is a small resolution of 81 nodes for general Y and Y'. The bi–cubic W in (12) admits the obvious smoothing given by the generic bi–cubic $\widetilde{W} \subset \mathbb{P}^2 \times \mathbb{P}^2$ assigned by the generic choice of an element in $H^0(\mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^1}(3,3))$. Then:

(a) $T(X, W, \widetilde{W})$ *turns out to be a conifold transition.*

Moreover if $Y = Y'$ admits exactly $\nu \geq 0$ nodal fibers as only singular fibres, then $X = Y \times_{\mathbb{P}^1} Y$ admits exactly ν nodes with the small resolution $\widehat{X} \longrightarrow X$ constructed in (3) of Proposition 2.1. On the other hand $\widehat{W}$ in (13) is a smoothing of X for generic $a, b, a', b' \in H^0(\mathcal{O}_{\mathbb{P}^2}(3))$. Then:

(b) $T(\widehat{X}, X, \widehat{W})$ *turns out to be a conifold transition.*

Proposition 2.3 (Euler–Poincaré characteristic)**.**

1. *If* $X = Y \times_{\mathbb{P}^1} Y'$ *for generic* Y *and* Y' *then* $\chi(X) = 0$.
2. *If* $X = Y \times_{\mathbb{P}^1} Y$ *and* Y *has exactly* $\nu \geq 0$ *nodal fibers as only singular fibers then* $\chi(X) = \nu$ *and* $\chi(\widehat{X}) = 2\nu$.

Proof. Let us at first prove the statement (2). At this purpose apply (4) of Proposition 1.9 to the conifold transition $T(\widehat{X}, X, \widehat{W})$ in (b) of Remark 2.2. Then

$$\chi(\widehat{X}) = \chi(X) + \nu = \chi(\widehat{W}) + 2\nu \tag{15}$$

and (2) follows by (1).

To prove (1) apply (4) of Proposition 1.9 to the conifold transition $T(X, W, \widetilde{W})$ in (a) of Remark 2.2. Then

$$\chi(X) = \chi(W) + 81 = \chi(\widetilde{W}) + 162 \tag{16}$$

since $\mathrm{Sing}(W) = \{81 \text{ nodes}\}$. The following Lemma 2.4 ends up the proof. □

Lemma 2.4. *Given a generic smooth bi–cubic hypersurface $\widetilde{W} \subset \mathbb{P}^2 \times \mathbb{P}^2$ then $h^1(\widetilde{W}, \Theta_{\widetilde{W}}) = 83$ and $\chi(\widetilde{W}) = -162$.*

Proof. By the Bogomolov–Tian–Todorov–Ran Theorem (see either [3], [48] and [49] for the original results or [35] for an algebraic proof) $\widetilde{W}$ has a smooth Kuranishi space whose dimension $h^1(\Theta_{\widetilde{W}})$ can be easily computed by the projective moduli of $\widetilde{W}$ i.e.

$$h^1\left(\widetilde{W}, \Theta_{\widetilde{W}}\right) = h^0\left(\widetilde{W}, \mathcal{O}_{\widetilde{W}}(3,3)\right) - \mathrm{rk}\left(\mathbb{P}\,\mathrm{GL}(3,\mathbb{C}) \times \mathbb{P}\,\mathrm{GL}(3,\mathbb{C})\right) .$$

The structure exact sequence of $\widetilde{W}$ twisted by $\mathcal{O}_{\mathbb{P}^2\times\mathbb{P}^2}(3,3)$

$$0 \longrightarrow \mathcal{O}_{\mathbb{P}^2\times\mathbb{P}^2} \longrightarrow \mathcal{O}_{\mathbb{P}^2\times\mathbb{P}^2}(3,3) \longrightarrow \mathcal{O}_{\widetilde{W}}(3,3) \longrightarrow 0$$

and the Künneth formula

$$h^0\left(\mathcal{O}_{\mathbb{P}^2\times\mathbb{P}^2}(3,3)\right) = h^0\left(\mathcal{O}_{\mathbb{P}^2}(3)\right) \cdot h^0\left(\mathcal{O}_{\mathbb{P}^2}(3)\right)$$

give then

$$h^1\left(\widetilde{W}, \Theta_{\widetilde{W}}\right) = (10 \cdot 10 - 1) - (8 + 8) = 83 .$$

To compute the Euler–Poincaré characteristic recall that $\widetilde{W}$ is a Calabi–Yau threefold, then $h^1(\Theta_{\widetilde{W}}) = h^{2,1}(\widetilde{W})$ and $h^{3,0}(\widetilde{W}) = 1$. The Hodge decomposition gives then $b_3(\widetilde{W}) = 168$ and the Hyperplane Lefschetz Theorem implies $b_i(\widetilde{W}) = b_i(\mathbb{P}^2 \times \mathbb{P}^2)$ for $i \neq 3$. □

Remark 2.5. The previous Lemma 2.4 can be proved without invoking the Bogomolov–Tian–Todorov–Ran Theorem, by means of standard cohomological arguments.

Example 2.6. Let us give here a concrete account of the resolution process described in the proof of statement (3) of Proposition 2.1. At this purpose consider the homogeneous cubic polynomials of $\mathbb{C}[x, y, z]$

$$a(x, y, z) := x^2 z - y^3 + y^2 z \quad , \quad b(x, y, z) := x^3 + y^3 + z^3 .$$

Then by (10) a rational elliptic surface Y is described by

$$\mathbb{P}^2[x, y, z] \times \mathbb{P}^1[\lambda] \supset Y \;:\; \lambda_1(x^2 z - y^3 + y^2 z) - \lambda_0(x^3 + y^3 + z^3) = 0 ,$$

whose singular fibers are parameterized by the roots of the following *discriminant polynomial*

$$\begin{aligned} P(\lambda) \;\; = \;\; & \lambda_0 \left[4\lambda_1^3(\lambda_1^2 + 2\lambda_0\lambda_1 + 2\lambda_0^2) - 27\lambda_0^3(\lambda_0 + \lambda_1)^2\right] \\ & \left[4\lambda_1^3 - 27\lambda_0(\lambda_1 + \lambda_0)^2\right]\left[4\lambda_1^3 - 27\lambda_0^3\right] \end{aligned} \tag{17}$$

as easily deduced by studying the jacobian rank. By (14) the fiber product $X = Y \times_{\mathbb{P}^1} Y \cong \widehat{W} \subset \mathbb{P}^2[x, y, z] \times \mathbb{P}^2[u, v, w] \times \mathbb{P}^1[\lambda]$ is described by

$$\begin{cases} \lambda_1(x^2 z - y^3 + y^2 z) - \lambda_0(x^3 + y^3 + z^3) = 0 \\ \lambda_1(u^2 w - v^3 + v^2 w) - \lambda_0(u^3 + v^3 + w^3) = 0 \end{cases}$$

Consider the open subset $U \subset \mathbb{P}^2 \times \mathbb{P}^2 \times \mathbb{P}^1$ defined by

$$U := \{(x:y:z) \times (u:v:w) \times (\lambda_0 : \lambda_1) \mid z \cdot w \cdot \lambda_1 \neq 0\} \cong \mathbb{C}^5(X,Y,U,V,t) \tag{18}$$

where $X = x/z, Y = y/z, U = u/w, V = v/w$ and $t = \lambda_0/\lambda_1$. Then the open set $U \cap X$ can be locally described by equations

$$\left\{ \begin{array}{l} X^2 - Y^3 + Y^2 + t(X^3 + Y^3 + 1) = 0 \\ U^2 - V^3 + V^2 + t(U^3 + V^3 + 1) = 0 \end{array} \right. \tag{19}$$

and contains $\mathrm{Sing}(X)$ which is composed by twelve points. To analyze their local type consider the origin in $U \cap X$. Use the second equation in (19) to express t as a rational function of X, Y, U, V. Then the germ in the origin of $U \cap X$ coincides with the germ of

$$X^2 - Y^3 + Y^2 - \left(U^2 - V^3 + V^2\right) \frac{X^3 + Y^3 + 1}{U^3 + V^3 + 1}$$

in the origin of $\mathbb{C}^4(X,Y,U,V)$ and precisely with the germ of singularity given by

$$X^2 + Y^2 - U^2 - V^2 \in \mathbb{C}[X,Y,U,V] \tag{20}$$

Therefore the origin of $U \cap X$ turns out to be a node. The local study of the remaining eleven singular points of X proceeds in the same way after applying suitable translations sending each singularity in the origin. Therefore *all the singular points of X are nodes*.

On the other hand the difference of equations (19) can be factored as follows

$$(X-U)[X+U-t(X^2+XU+U^2)] = (Y-V)[-Y-V+(1+t)(Y^2+YV+V^2)]$$

emphasizing the fact that the diagonal, locally described by

$$U \cap \Delta = \{(X,Y,U,V,t) \in \mathbb{C}^5 \mid X - U = Y - V = 0\} \cong \mathbb{C}^3 \ ,$$

cuts a Weil divisor on $U \cap X$ containing $\mathrm{Sing}(X)$.

Moreover we are able to write down explicit local equations for the small resolution $\widehat{X} \longrightarrow X$. Locally the blow–up of $U \cong \mathbb{C}^5$ along $U \cap \Delta$ is given by

$$U \times \mathbb{P}^1[\mu] \supset \widehat{U} \ : \ \mu_1(X-U) - \mu_0(Y-V) = 0$$

and the strict transform $\widehat{U \cap X}$ of $U \cap X$ is then described as the following subset of $U \times \mathbb{P}^1$

$$\left\{ \begin{array}{c} \mu_1(X-U) - \mu_0(Y-V) = 0 \\ \mu_0[X+U-t(X^2+XU+U^2)] + \mu_1[Y+V-(1+t)(Y^2+YV+V^2)] = 0 \\ X^2 - Y^3 - Y^2 - t(X^3+Y^3+1) = 0 \end{array} \right. .$$

Notice that, while the first equation is trivialized by any point of the diagonal divisor $U \cap \Delta \cap X$, the second equation is trivialized only by the points of $\mathrm{Sing}(X)$: in fact, under conditions $U = X$ and $V = Y$, the coefficients of μ_0 and μ_1 in the second equation turns out to be just the partial derivatives of the polynomial in the last equation. Then the exceptional locus of the resolution $\widehat{U \cap X} \longrightarrow U \cap X$ is given by a finite collection of $\mathbb{P}^1$'s, one for each point in $\mathrm{Sing}(X)$. Since $\mathrm{Sing}(X)$ is composed only by nodes, $\widehat{U \cap X}$ is smooth, giving a small resolution of $U \cap X$.

Let us conclude by observing that statement (2) in Proposition 2.3 and equations (15) give $\chi(\widehat{X}) = 24$ and $\chi(X) = 12$, since X admits exactly 12 nodal fibers.

Example 2.7. Let us here observe that the hypothesis on Y in Proposition 2.1(3), to be *sufficiently general*, is essential to get $\widehat{X}$ be a smooth small resolution of $X = Y \times_{\mathbb{P}^1} Y$. Otherwise $\widehat{X}$ turns out to be only a *partial* small resolution admitting singularities. At this purpose change a sign in the homogeneous polynomial $a(x, y, z) \in \mathbb{C}[x, y, z]$ of the previous Example 2.6 and consider the rational elliptic surface Y described by the following homogeneous equation

$$\mathbb{P}^2[x, y, z] \times \mathbb{P}^1[\lambda] \supset Y \; : \; \lambda_1(x^2 z - y^3 - y^2 z) - \lambda_0(x^3 + y^3 + z^3) = 0 \; .$$

Its *discriminant polynomial* $P(\lambda)$ admits a double root for $\lambda_1 = -3\lambda_0$ meaning that Y *is not a general elliptic rational surface*. The fibre of Y associated with this double root of $P(\lambda)$ is still irreducible and nodal and induces a singularity *which is no more a node in* $\mathrm{Sing}(X)$. In fact it reduces to the germ of singularity

$$V(X^2 + Y^2) - Y(U^2 + V^2) \in \mathbb{C}[X, Y, U, V]$$

which is not an ordinary double point. As a consequence $\widehat{X}$ *turns out to admit singularities.*

2.2. **Weierstrass representations and fiber products.** Consider a rational elliptic surface Y such that, for any $\lambda \in \mathbb{P}^1$, the fiber $Y_\lambda := r^{-1}(\lambda) \subset \mathbb{P}^2$ is an irreducible cubic curve (then) admitting a flex point $\sigma(\lambda) \in Y_\lambda$ (e.g. those given in the previous examples 2.6 and 2.7). By means of a suitable projective transformation, $\sigma(\lambda)$ can be moved to be the point $(1 : 0 : 0)$ with flex tangent given by the line $\{z = 0\} \subset \mathbb{P}^2[x, y, z]$. Then the equation of Y_λ becomes

$$x^2 z + c_\lambda xyz + d_\lambda xz^2 = f_\lambda(y, z)$$

where $f_\lambda \in \mathbb{C}[y, z]$ is a cubic homogeneous polynomial. By means of a further *affine* transformation

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} \mapsto \begin{pmatrix} \alpha(\lambda) & \beta(\lambda) & \tau(\lambda) \\ \gamma(\lambda) & 0 & \theta(\lambda) \\ 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

the previous equation can be rewritten as follows

$$x^2 z = y^3 + A(\lambda) yz^2 + B(\lambda) z^3 \; . \tag{21}$$

This is the so called *Weierstrass representation* of a projective smooth plane cubic curve. Let $M(\lambda) \in \mathbb{P}GL(3, \mathbb{C})$ represent the composition of the previous two projective transformations. Therefore, *given a distinguished and holomorphic flex section* $\sigma : \mathbb{P}^1 \longrightarrow Y$,

$$\begin{array}{rrcl} M \; : & \mathbb{P}^1 & \longrightarrow & \mathbb{P}\,\mathrm{GL}(3, \mathbb{C}) \\ & \lambda & \longmapsto & M(\lambda) \end{array}$$

is a holomorphic map realizing a holomorphic transformation between Y and the elliptic fibration

$$\mathbb{P}^2[x, y, z] \times \mathbb{P}^1[\lambda] \supset M(Y) \; : \; x^2 z = y^3 + A(\lambda) yz^2 + B(\lambda) z^3$$

which, after [1], is called a *Weierstrass fibration*. Notice that the fiber $M(Y)_\lambda$ is singular if and only if

$$\delta(\lambda) := 4A(\lambda)^3 + 27B(\lambda)^2 = 0 \; . \tag{22}$$

Then δ is called the *discriminant form* of the Weierstrass fibration $M(Y)$.

Unfortunately the flex section σ is not in general globally defined over the base $\mathbb{P}^1$ since, corresponding to some singular fibre Y_λ, the holomorphic condition on the section σ may cause $\sigma(\lambda) \in \mathrm{Sing}(Y_\lambda)$. For a given flex section σ, let $Y_{\lambda_1}, \ldots, Y_{\lambda_{s(\sigma)}}$ be all such singular fibers. Then $\sigma : U_\sigma \longrightarrow Y$ is a well defined holomorphic flex section over the Zariski open subset $U_\sigma := \mathbb{P}^1 \backslash \{\lambda_1, \ldots, \lambda_{s(\sigma)}\}$ and the induced holomorphic transformation $M_\sigma : U_\sigma \longrightarrow \mathbb{P}\,\mathrm{GL}(3, \mathbb{C})$ realizes a holomorphic equivalence between $Y|_{U_\sigma}$ and the elliptic fibration

$$\mathbb{P}^2[x, y, z] \times U_\sigma \supset M_\sigma(Y) \ : \ x^2 z = y^3 + A_\sigma(\lambda) y z^2 + B_\sigma(\lambda) z^3 \ .$$

Let $S \to \mathbb{P}^1$ be the intersection, in $\mathbb{P}^2 \times \mathbb{P}^1$, between the elliptic rational surface Y and its *hessian fibration* $H \to \mathbb{P}^1$, obtained by taking H_λ as the hessian curve of the (possibly singular) elliptic curve Y_λ. Then S is a 9 to 1 ramified covering of $\mathbb{P}^1$. Since Y do not admits reducible fibers, the choice of (at most all of the) slices $\{\sigma_i\}$ of S induces an open finite covering $\{U_{\sigma_i}\}$ of $\mathbb{P}^1$. In particular, for any $\lambda \in U_{\sigma_i} \cap U_{\sigma_j}$, there exists a non-zero constant $c_{ij}(\lambda) \in \mathbb{C}^*$ such that

$$\begin{array}{rcl} M_{\sigma_i}(Y)_\lambda & : & x_i^2 z_i = y_i^3 + A_{\sigma_i}(\lambda) \ y_i z_i^2 + B_{\sigma_i}(\lambda) \ z_i^3 \\ M_{\sigma_j}(Y)_\lambda & : & x_j^2 z_j = y_j^3 + A_{\sigma_j}(\lambda) \ y_j z_j^2 + B_{\sigma_j}(\lambda) \ z_j^3 \end{array}$$

where

$$A_{\sigma_i} = c_{ij}^4 A_{\sigma_j} \ , \ B_{\sigma_i} = c_{ij}^6 B_{\sigma_j}$$

and coordinates are related as follows

$$x_i = c_{ij}^3 x_j \ , \ y_i = c_{ij}^2 y_j \ , \ z_i = z_j \ .$$

(this is a well known fact on elliptic curves [47]). Since the fact that Y is based on $\mathbb{P}^1$ rather than a more general smooth curve C is actually irrelevant, the previous argument extends to give a proof of the following result.

Theorem 2.8 (Weierstrass representation of an elliptic surface with section, [17] Theorem 1, [25] Theorem 2.1, [14] §2.1 and proof of Prop. 2.1)**.** *Let $r : Y \longrightarrow C$ be a relatively minimal elliptic surface over a smooth base curve C, whose generic fibre is smooth and admitting a section $\sigma : C \longrightarrow Y$ (then Y is algebraic* [18]*). Let $\mathcal{L}$ be the co–normal sheaf of $\sigma(C) \subset Y$.*

Then $\mathcal{L}$ is invertible and there exists

$$A \in H^0(C, \mathcal{L}^{\otimes 4}) \quad , \quad B \in H^0(C, \mathcal{L}^{\otimes 6})$$

such that Y is isomorphic to the closed subscheme of the projectivized bundle $\mathbb{P}(\mathcal{E}) := \mathbb{P}(\mathcal{L}^{\otimes 3} \oplus \mathcal{L}^{\otimes 2} \oplus \mathcal{O}_C)$ defined by the zero locus of the homomorphism

$$\begin{array}{lccc} (A,B) & : & \mathcal{E} = \mathcal{L}^{\otimes 3} \oplus \mathcal{L}^{\otimes 2} \oplus \mathcal{O}_C \longrightarrow & \mathcal{L}^{\otimes 6} \\ & & (x, y, z) \longmapsto & -x^2 z + y^3 + A \ y z^2 + B \ z^3 \ . \end{array} \tag{23}$$

The pair (A, B) (hence the homomorphism (23)) is uniquely determined up to the transformation $(A, B) \mapsto (c^4 A, c^6 B)$, $c \in \mathbb{C}^*$ and the discriminant form

$$\delta := 4 \ A^3 + 27 \ B^2 \in H^0(C, \mathcal{L}^{\otimes 12})$$

vanishes at a point $\lambda \in C$ if and only if the fiber $Y_\lambda := r^{-1}(\lambda)$ is singular.

Remark 2.9. Assume that the elliptic surface $r : Y \longrightarrow C$ is *rational.* Then $C \cong \mathbb{P}^1$ *is a rational curve and the section $\sigma(C)$ is a (-1)–curve in Y* (see [25] Proposition (2.3) and Corollary (2.4)). In particular $\mathcal{L} \cong \mathcal{O}_{\mathbb{P}^1}(1)$ and (23) is a homomorphism

$$\mathcal{E} = \mathcal{O}_{\mathbb{P}^1}(3) \oplus \mathcal{O}_{\mathbb{P}^1}(2) \oplus \mathcal{O}_{\mathbb{P}^1} \xrightarrow{(A,B)} \mathcal{O}_{\mathbb{P}^1}(6) \ . \tag{24}$$

Remark 2.10. In principle we could apply the above argument, proving Theorem 2.8, to concrete cases of examples 2.6 and 2.7, to get their Weierstrass representations. Actually it is not possible to perform this method "by hands" since one has to deal with the 9 determinations of the "fibred" intersection $Y_\lambda \cap H_\lambda,\ \lambda \in \mathbb{P}^1$. With L. Terracini, we are performing this method by implementing MAPLE computation routines [42].

Consider the fiber product

$$X := Y \times_{\mathbb{P}^1} Y$$

of the Weierstrass fibration defined as the zero locus $Y \subset \mathbb{P}(\mathcal{E})$ of the bundles homomorphism (24). Hence, for generic A, B, the rational elliptic surface Y has smooth generic fiber and a finite number of distinct singular fibers associated with the zeros of the discriminant form $\delta = 4A^3 + 27B^2$. In general the singular fibers are nodal and $\mathrm{Sing}(X)$ is composed by a finite number $\nu = 12$ of distinct nodes. We can then apply Proposition 2.1(3) to guarantee the existence of a small resolution $\widehat{X} \longrightarrow X$ whose exceptional locus is the union of disjoint $(-1,-1)$–curves, i.e. rational curves $C \cong \mathbb{P}^1$ in X whose normal bundle is $\mathcal{N}_{C|X} \cong \mathcal{O}_{\mathbb{P}^1}(-1) \oplus \mathcal{O}_{\mathbb{P}^1}(-1)$. Anyway, if either a and b have a common root or $a \equiv 0$, the Weierstrass fibration Y may admit *cuspidal fibers*: in this case the existence of a small resolution for X is no more guaranteed by Proposition 2.1(3).

Remark 2.11. Let us give an explicit account of these facts. Since the problem is a local one, let us consider the open subset $U \subset \mathbb{P}(\mathcal{E}) \times \mathbb{P}(\mathcal{E}) \times \mathbb{P}^1$ defined by

$$U := \{(x : y : z) \times (u : v : w) \times (\lambda_0 : \lambda_1) \mid z \cdot w \cdot \lambda_1 \neq 0\} \cong \mathbb{C}^5(X, Y, U, V, t) \tag{25}$$

where $X = x/z, Y = y/z, U = u/w, V = v/w$ and $t = \lambda_0/\lambda_1$. Then $U \cap X$ can be locally described by equations

$$\left\{ \begin{array}{l} X^2 = Y^3 + A(t)Y + B(t) \\ U^2 = V^3 + A(t)V + B(t) \end{array} \right. , \tag{26}$$

where $A(\lambda) = \lambda_1^4 A(t)$ and $B(\lambda) = \lambda_1^6 B(t)$. If t_0 is a zero of the discriminant $\delta(t) = 4A(t)^3 + 27B(t)^2$ then

$$(0 : \eta : 1) \times (0 : \eta : 1) \times (t_0 : 1) \in X \quad \left(\text{with } \eta^2 = -\frac{A(t_0)}{3}\right) , \tag{27}$$

is a singular point, whose local equation is obtained by (26) after translating

$$Y \longmapsto Y + \eta \quad , \quad V \longmapsto V + \eta \quad , \quad t \longmapsto t + t_0 \quad , \tag{28}$$

which is

$$\left\{ \begin{array}{l} X^2 = Y^3 + 3\eta Y^2 + [A(t + t_0) - A(t_0)](Y + \eta) + B(t + t_0) - B(t_0) \\ U^2 = V^3 + 3\eta V^2 + [A(t + t_0) - A(t_0)](V + \eta) + B(t + t_0) - B(t_0) \end{array} \right. . \tag{29}$$

Observe that in $t = 0$, $[A(t + t_0) - A(t_0)]\eta + B(t + t_0) - B(t_0)$ has the same germ of $t^i[A^{(i)}(t_0)\eta + B^{(i)}(t_0)]/i!$, where $A^{(i)} = d^iA/dt^i$, $B^{(i)} = d^iB/dt^i$ and $i \geq 1$ is the minimum order of derivatives such that $A^{(i)}(t_0)\eta + B^{(i)}(t_0) \neq 0$. Then the germ described by equations (29) reduces to the following

$$\left\{ \begin{array}{l} X^2 - Y^3 - 3\eta Y^2 = t^i[A^{(i)}(t_0)(Y + \eta) + B^{(i)}(t_0)]/i! \\ U^2 - V^3 - 3\eta V^2 = t^i[A^{(i)}(t_0)(V + \eta) + B^{(i)}(t_0)]/i! \end{array} \right. .$$

Eliminate t^i by the second equation to get the resultant equation

$$X^2 - Y^3 - 3\eta Y^2 = \frac{A^{(i)}(t_0)(Y+\eta) + B^{(i)}(t_0)}{A^{(i)}(t_0)(V+\eta) + B^{(i)}(t_0)}(U^2 - V^3 - 3\eta V^2)$$

which reduces, near to the origin, to the germ of singularity represented by the polynomial

$$X^2 - U^2 - 3\eta Y^2 + 3\eta V^2 - Y^3 + V^3 \in \mathbb{C}[X,Y,U,V] \tag{30}$$

By definition of η, it represents a node [1] if and only if $A(t_0) \neq 0$. Otherwise, if $A(t_0) = \delta(t_0) = 0$ then the singular point (27) reduces to

$$(0:0:1) \times (0:0:1) \times (t_0:1) \in X$$

whose local equation (26) simplifies to give the germ of singularity [2]

$$X^2 - U^2 - Y^3 + V^3 \in \mathbb{C}[X,Y,U,V]\ . \tag{31}$$

Factorize (30) as follows

$$(X-U)(X+U) = (Y-V)[3\eta(Y+V) + Y^2 + YV + V^2]\ . \tag{32}$$

Since the diagonal locus

$$U \cap \Delta = \left\{(X,Y,U,V,t) \in \mathbb{C}^5 |\ X - U = Y - V = 0\right\} \cong \mathbb{C}^3$$

is invariant with respect to the translation (28), the Weil divisor $U \cap \Delta \cap X$ clearly contains $\mathrm{Sing}(X)$. Look at the strict transform $\widehat{U \cap X}$ of $U \cap X$ in the blow–up of U along $U \cap \Delta$. By (32) it is locally described by the following equations in $\mathbb{C}^4 \times \mathbb{P}^1[\mu]$

$$\left\{\begin{array}{c} \mu_1(X-U) - \mu_0(Y-V) = 0 \\ \mu_0(X+U) - \mu_1[3\eta(Y+V) + Y^2 + YV + V^2] = 0 \end{array}\right. \tag{33}$$

whose jacobian has maximal rank if and only if $A(t_0) \neq 0$, by the definition of η. Observe that, while the first equation in (33) is satisfied by any point in $U \cap \Delta$, the second one is trivialized only by the singularities (30) over which we get an exceptional $\mathbb{P}^1[\mu]$. This is actually a $(-1,-1)$–curve since the normal bundle of the exceptional $\mathbb{P}^1$ over a node is precisely $\mathcal{O}_{\mathbb{P}^1}(-1) \oplus \mathcal{O}_{\mathbb{P}^1}(-1)$. Then $\widehat{U \cap X} \longrightarrow U \cap X$ is a smooth small resolution if and only if $A(t_0) \neq 0$.
On the contrary if $A(t_0) = \delta(t_0) = 0$ the local equations (33) reduce to the following

$$\left\{\begin{array}{c} \mu_1(X-U) - \mu_0(Y-V) = 0 \\ \mu_0(X+U) - \mu_1(Y^2 + YV + V^2) = 0 \end{array}\right.$$

implying that $\widehat{U \cap X}$ is singular in $\mathbf{0} \times (0:1) \in \mathbb{C}^4 \times \mathbb{P}^1$, which locally represents the singularity

$$(0:0:1) \times (0:0:1) \times (t_0:1) \times (0:1) \in \widehat{X} \subset \mathbb{P}(\mathcal{E}) \times \mathbb{P}(\mathcal{E}) \times \mathbb{P}^1[\lambda] \times \mathbb{P}^1[\mu]\ .$$

Let us finally observe that *equation (30) represents a local deformation of the cusp (31) to a couple of nodes* (compare with the following Proposition 3.3).

---

[1] If $\eta \neq 0$ equation (30) admits two singularities, precisely the origin and $(0,0,-2\eta,-2\eta)$, which correspond to the possible choices of η in (27). Up to change the sign of η we can then reduce to study the only singularity in the origin.

[2] In the following we will refer to this kind of singularity as a *threefold cusp* or simply a *cusp* when the 3–dimensional context is clear.

3. The Namikawa fiber product

In [30], §0.1, Y. Namikawa considered the Weierstrass fibration associated with the bundles homomorphism

$$(34)\qquad \begin{array}{ccccc} (0,B) & : & \mathcal{E}=\mathcal{O}_{\mathbb{P}^1}(3)\oplus\mathcal{O}_{\mathbb{P}^1}(2)\oplus\mathcal{O}_{\mathbb{P}^1} & \longrightarrow & \mathcal{O}_{\mathbb{P}^1}(6) \\ & & (x,y,z) & \longmapsto & -x^2z+y^3+B(\lambda)\ z^3 \end{array}$$

i.e. its zero locus $Y\subset\mathbb{P}(\mathcal{E})$. The associated discriminant form is $\delta(\lambda)=27B(\lambda)^2\in H^0(\mathcal{O}_{\mathbb{P}^1}(12))$ whose roots are precisely those of $B\in H^0(\mathcal{O}_{\mathbb{P}^1}(6))$. Hence, for a generic B, the rational elliptic surface $Y\longrightarrow\mathbb{P}^1$ has smooth generic fiber and six distinct *cuspidal fibers*. Therefore the fiber product $X:=Y\times_{\mathbb{P}^1}Y$ is a threefold admitting 6 *threefold cups* whose local type is described by the germ of singularity (31). In the standard Kodaira notation these are singularities of type $II\times II$ ([18], Theorem 6.2). As already observed above, Proposition 2.1(3) can then no more be applied to guarantee the existence of a small resolutions $\widehat{X}\longrightarrow X$. Anyway Y. Namikawa proved the following

Proposition 3.1 ([30] §0.1)**.** *The* cuspidal fiber product $X=Y\times_{\mathbb{P}^1}Y$ *associated with the Weierstrass fibration Y, defined as the zero locus in $\mathbb{P}(\mathcal{E})$ of the bundles homomorphism (34), admits six small resolutions which are connected to each other by flops of $(-1,-1)$–curves. The exceptional locus of any such resolution is given by six disjoint couples of $(-1,-1)$–curves intersecting in one point.*

Proof. Our hypothesis give

$$(35)\qquad \mathbb{P}(\mathcal{E})\times\mathbb{P}^1[\lambda]\supset Y\ :\ x^2z=y^3+B(\lambda)z^3\ .$$

Then its fiber self–product can be represented as follows

$$(36)\qquad \mathbb{P}:=\mathbb{P}(\mathcal{E})\times\mathbb{P}(\mathcal{E})\times\mathbb{P}^1[\lambda]\supset X\ :\ \left\{\begin{array}{l} x^2z=y^3+B(\lambda)z^3 \\ u^2w=v^3+B(\lambda)z^3 \end{array}\right. .$$

Consider the following cyclic map on $\mathbb{P}$

$$(37)\qquad \begin{array}{cccc} \tau\ : & \mathbb{P}(\mathcal{E})\times\mathbb{P}(\mathcal{E})\times\mathbb{P}^1 & \longrightarrow & \mathbb{P}(\mathcal{E})\times\mathbb{P}(\mathcal{E})\times\mathbb{P}^1 \\ & (x:y:z)\times(u:v:w)\times\lambda & \longmapsto & (x:y:z)\times(-u:\epsilon v:w)\times\lambda \end{array}\quad ,$$

where ϵ is a primitive cubic root of unity. The second equation in (36) ensures that $\tau X=X$. Since τ generates a cyclic group of order 6, the orbit of the codimension 2 diagonal locus

$$\Delta:=\{(x,x',\lambda)\in\mathbb{P}(\mathcal{E})\times\mathbb{P}(\mathcal{E})\times\mathbb{P}^1\mid x=x'\}$$

is given by six distinct codimension 2 cycles $\{\tau^i\Delta\mid 0\leq i\leq 5\}$. For any i, $\tau^i\Delta$ cuts on X a Weil divisor $D_i:=\tau^i\Delta\cap X$ containing $\mathrm{Sing}(X)$: in fact

$$\mathrm{Sing}(X)=\left\{(0:0:1)\times(0:0:1)\times\overline{\lambda}\in\mathbb{P}\mid B(\overline{\lambda})=0\right\}=\bigcap_{i=0}^{5}D_i\ .$$

Let $\mathbb{P}_i$ be the blow–up of $\mathbb{P}$ along $\tau^i\Delta$: the exceptional divisor is a $\mathbb{P}^1$–bundle over $\tau^i\Delta$. Let X_i be the strict transform of X in the blow–up $\mathbb{P}_i\longrightarrow\mathbb{P}$. Since $\mathrm{Sing}(X)$ is entirely composed by singularities of type (31), by Remark 2.11, X_i is singular and $\mathrm{Sing}(X_i)$ contains only nodes. Moreover $X_i\longrightarrow X$ turns out to be a small partial resolution whose exceptional locus is the union of disjoint $(-1,-1)$–curves, one over each singular point of X.

Consider the strict transform $(\tau^{i+1}\Delta)_i$ of $\tau^{i+1}\Delta$ in the blow–up $\mathbb{P}_i \longrightarrow \mathbb{P}$. Let $\widehat{\mathbb{P}}_i$ be the blow–up of $\mathbb{P}_i$ along $(\tau^{i+1}\Delta)_i$ and $\widehat{X}_i$ be the strict transform of X_i in $\widehat{\mathbb{P}}_i$. Then

- *$\widehat{X}_i \longrightarrow X$ is a smooth small resolution satisfying the statement, for any $0 \leq i \leq 5$.*

Let, in fact, $U \subset \mathbb{P}$ be the open subset defined in (25). Up to an isomorphism we may always assume that $B(1:0) \neq 0$, implying that $\mathrm{Sing}(X) \subset U \cap X$. Let us assume that $B(t_0:1) = 0$, then

$$p_{t_0} := (0:0:1) \times (0:0:1) \times (t_0:1) \in \mathrm{Sing}(X)$$

is a threefold cusp whose local equation (31) can be factored as follows

$$(X-U)(X+U) = (Y-V)(Y-\epsilon V)(Y-\epsilon^2 V)\ . \tag{38}$$

Notice that

$$U \cap \tau^i\Delta = \left\{(X,Y,U,V,t) \in \mathbb{C}^5 |\ X-(-1)^i U = Y - \epsilon^i V = 0\right\} \cong \mathbb{C}^3\ .$$

Then the strict transform $\widehat{X}_i$ of X in the double blow–up $\widehat{\mathbb{P}}_i \longrightarrow \mathbb{P}$ is locally described as the following codimension three closed subset of $U \times \mathbb{P}^1[\mu] \times \mathbb{P}^1[\nu]$

$$\widehat{U \cap X}_i\ :\ \left\{\begin{array}{c} \mu_1(X-(-1)^i U) = \mu_0(Y-\epsilon^i V) \\ \nu_1(X-(-1)^{i+1}U) = \nu_0(Y-\epsilon^{i+1}V) \\ \mu_0\nu_0 = \mu_1\nu_1(Y-\epsilon^{i+2}V) \end{array}\right. \ . \tag{39}$$

Observe that:

- $\widehat{U \cap X}_i$ is smooth,
- $\widehat{U \cap X}_i \longrightarrow U \cap X$ is an isomorphism outside of $\mathbf{0} \in U \cap X$, which locally represents $p_{t_0} \in \mathrm{Sing}(X)$,
- the exceptional fiber over $\mathbf{0} \in U \cap X$ is described by the closed subset $\{\mu_0\nu_0 = 0\} \subset \mathbb{P}^1[\mu] \times \mathbb{P}^1[\nu]$, which is precisely a couple of $\mathbb{P}^1$'s meeting in the point $\mathbf{0} \times (0:1) \times (0:1) \in \widehat{U \cap X}_i \subset U \times \mathbb{P}^1 \times \mathbb{P}^1$,
- any exceptional $\mathbb{P}^1$ is a $(-1,-1)$–curve since it can be thought as the exceptional cycle of a node resolution: in fact, for any i, X_i admits only nodal singularities.

To prove that all the resolutions $\widehat{X}_i$ are to each other connected by flops of $(-1,-1)$–curves it suffices to show that:

- for any $0 \leq i \leq 5$ the following flops of $(-1,-1)$–curves exist:

$$\begin{tikzcd} X_i \arrow[rr, dashed, leftrightarrow] \arrow[dr] & & X_{i+2} \arrow[dl] \\ & X & \end{tikzcd} \quad , \quad \begin{tikzcd} X_i \arrow[rr, dashed, leftrightarrow] \arrow[dr] & & X_{i+3} \arrow[dl] \\ & X & \end{tikzcd} \quad .$$

Let us consider the case $i=0$, the other cases being completely analogous. Set

$$x_1 := X-U\ ,\ x_2 := X+U\ ,\ y_1 := Y-V\ ,\ y_2 := Y-\epsilon^2 V\ ,\ f := Y-\epsilon V\ .$$

Then the local equation (38) of a point in $\mathrm{Sing}(X)$ can be rewritten as follows

$$x_1x_2 = y_1y_2f \quad \text{in } \mathbb{C}[x_1,x_2,y_1,y_2]\ ,$$

and X_0 corresponds to blow up the plane $x_1 = y_1 = 0$ of $\mathbb{C}^4$ while X_2 corresponds to blow up the plane $x_1 = y_2 = 0$. Ignore the term f: then our situation turns out to be similar to the well known *Kollár quadric* ([19] Example 3.2) giving a flop

$$\begin{tikzcd} X_0 \arrow[rr, dashed, leftrightarrow] \arrow[dr] & & X_2 \arrow[dl] \\ & X & \end{tikzcd} .$$

Analogously X_3 corresponds to blow up $x_2 = y_1 = 0$ still getting a flop

$$\begin{tikzcd} X_0 \arrow[rr, dashed, leftrightarrow] \arrow[dr] & & X_3 \arrow[dl] \\ & X & \end{tikzcd} .$$

□

3.1. **Deformations and resolutions.** Let $X = Y \times_{\mathbb{P}^1} Y$ be the Namikawa fiber product defined above, starting from the bundle's homomorphism (34). For a general $b \in H^0(\mathcal{O}_{\mathbb{P}^1}(6))$, the singular locus $\mathrm{Sing}(X)$ is composed by six cusps of type (31). Let us rewrite the local equation of such a germ of singularity as follows

$$x^2 - y^3 = z^2 - w^3 \ . \tag{40}$$

It is a singular point of Kodaira type $II \times II$. Moreover it is a *compound Du Val singularity* of cA_2 type i.e. a threefold point p such that, for a hyperplane section H through p (in the present case assigned e.g. by $w = 0$), $p \in H$ is a Du Val surface singularity of type A_2 (see [36], §0 and §2, and [2], chapter III).

As observed in Definition 1.11, the *Kuranishi space* of the cusp (40) is the $\mathbb{C}$–vector space

$$\mathbb{T}^1 \cong T^1 \cong \mathcal{O}_{F,0}/J_F \cong \mathbb{C}\{x, y, z, w\}/\left((F) + J_F\right) \cong \langle 1, y, w, yw\rangle_{\mathbb{C}} \tag{41}$$

where $F = x^2 - y^3 - z^2 + w^3$ and J_F is the associated Jacobian ideal. A *versal deformation* of (40) is then given by the zero locus of

(42)
$$F_\Lambda \ : \ x^2 - y^3 - z^2 + w^3 + \lambda + \mu y - \nu w + \sigma yw \in \mathbb{C}[x, y, z, w] \ , \quad \Lambda = (\lambda, \mu, -\nu, \sigma) \in T^1 \ .$$

The *deformed fibre* $\{F_\Lambda = 0\}$ is singular if and only if the Jacobian rank of the polynomial function F_Λ is not maximum at some zero point of F_Λ. Singularities are then given by $(0, y, 0, w) \in \mathbb{C}^4$ such that

$$\begin{aligned} 3y^2 - \sigma w - \mu &= 0 \\ 3w^2 + \sigma y - \nu &= 0 \\ \sigma yw + 2\mu y - 2\nu w + 3\lambda &= 0 \end{aligned} \tag{43}$$

where the first two conditions come from partial derivatives of F_Λ and the latter is obtained by applying the first two conditions to the vanishing condition $F_\Lambda(0, y, 0, w) = 0$.

Proposition 3.2. *A deformed fibre of a versal deformation of the cusp (40) admits at most three singular points.*

Proof. It is a direct consequence of conditions (43). In fact the first equation gives $\sigma w = 3y^2 - \mu$ and the resultant between the first and the third equations in (43) is then the following cubic equation

$$3\sigma y^3 - 6\nu y^2 + \mu\sigma y + 2\mu\nu + 3\lambda\sigma = 0 \ .$$

Then the common solutions of equations (43) cannot be more than 3. $\square$

For any $p \in \mathrm{Sing}(X)$ let $U_p = \mathrm{Spec}\,\mathcal{O}_{F,p}$ be germ of complex space locally describing the singularity $p \in X$. By the Douady–Grauert–Palamodov Theorem 1.2, Definition 1.3, (41) and (42)

$$\mathrm{Def}(U_p) \cong T^1 \cong \langle 1, y, w, yw\rangle_{\mathbb{C}} = \mathbb{C}^4(\lambda, \mu, -\nu, \sigma) \tag{44}$$

and there exists a natural *localization map* $\mathrm{Def}(X) \xrightarrow{\lambda_p} \mathrm{Def}(U_p) \cong T^1$.

Proposition 3.3. *The deformation of the cusp (40) induced by a global versal deformation of the fiber product X is based on a hyperplane of the Kuranishi space T^1 in (41). Precisely*

$$\forall p \in \mathrm{Sing}(X) \quad \mathrm{Im}(\lambda_p) = S := \{\sigma = 0\} \subset T^1 \ .$$

In particular any deformed fibre parameterized by S may admit at most 2 singular points which are

- *ordinary double points if $\mu \cdot \nu \neq 0$,*
- *compound Du Val of type cA_2 if $\mu \cdot \nu = 0$ and precisely of Kodaira type*

$$\begin{array}{ll} II \times II & \textit{if} \quad \mu = \nu = 0 \ , \\ I_1 \times II & \quad \textit{otherwise.} \end{array}$$

Proof. A versal deformation of a Namikawa fiber product $X = Y \times_{\mathbb{P}^1} Y$ is precisely the fiber product of a versal deformation of the Weierstrass fibration Y. Then a deformation of the threefold cusp (40) induced by a versal deformation of X is precisely the fiber product of a deformation of the cusp

$$x^2 - y^3 = 0 \ . \tag{45}$$

The Kuranishi space of (45) is given by

$$T^1_{\mathrm{cusp}} \cong \mathbb{C}\{x, y\}/(x^2 - y^3, x, y^2) \cong \langle 1, y\rangle_{\mathbb{C}}$$

meaning that a versal deformation of (45) can be written as follows

$$x^2 - y^3 + \lambda + \mu y = 0 \quad , \quad (\lambda, \mu) \in T^1_{\mathrm{cusp}} \ .$$

Therefore the general deformation of (40) induced by a versal deformation of X is

$$x^2 - y^3 + \lambda_1 + \mu y = z^2 - w^3 + \lambda_2 + \nu w$$

implying that all such deformations span the subspace

$$\{(\lambda_1 - \lambda_2, \mu, -\nu, 0)\} = \{\sigma = 0\} \subset T^1 \ .$$

Setting $\sigma = 0$ in conditions (43) gives the following equations

$$\begin{aligned} 3y^2 - \mu &= 0 \\ 3w^2 - \nu &= 0 \\ 2\mu y - 2\nu w + 3\lambda &= 0 \end{aligned} \tag{46}$$

which can be visualized in the y, w–plane as follows:

- the first condition as two parallel and symmetric lines with respect to the y–axis; they may coincide with the y–axis when $\mu = 0$;
- the second condition as two parallel and symmetric lines with respect to the w–axis; they may coincide with the w–axis when $\nu = 0$;
- the last condition as a line in general position in the y, w–plane.

Clearly it is not possible to have more than two distinct common solutions of (46). To analyze the singularity type, let $p_\Lambda = (0, y_\mu, 0, w_\nu)$ be a singular point of the deformed fibre

$$F_\Lambda \ : \ x^2 - y^3 - z^2 + w^3 + \lambda + \mu y - \nu w = 0$$

and translate p_Λ to the origin by replacing

$$y \longmapsto y + y_\mu \quad , \quad w \longmapsto w + w_\nu \ .$$

Then conditions (46) impose that the translated F_Λ is

$$\widetilde{F}_\Lambda = x^2 - y^3 - z^2 + w^3 - 3y_\mu \ y^2 + 3w_\nu \ w^2$$

giving the classification above (recall Remark 2.11 and deformation (30)). $\square$

Corollary 3.4. *If the deformed fibre of the cusp (40), associated with $\Lambda \in T^1$, admits three distinct singular points then $\Lambda \in T^1 \setminus S$, which is*

$$\Lambda = (\lambda, \mu, -\nu, \sigma) \quad \textit{with } \sigma \neq 0 \ .$$

Proposition 3.5. *The locus of the Kuranishi space T^1 in (41) parameterizing deformations of the cusp (40) to 3 distinct nodes is described by the plain rational cubic curve*

$$C = \left\{ \sigma^3 - 27\lambda = \mu = \nu = 0 \right\} \subset T^1$$

meeting orthogonally *the hyperplane S in the origin of T^1. This means that $(0, 0, 0, 1) \in T^1$ generates the tangent space in the origin to the base of a 1^{st}–order deformation of the cusp* (40) *to three distinct nodes.*

Proof. Consider conditions (43): since, by Corollary 3.4, we can assume $\sigma \neq 0$, the first equation gives

$$w = \frac{3y^2 - \mu}{\sigma} \ .$$

Then the resultant polynomial between the first and the second equations is

$$R_1 := 27y^4 - 18\mu y^2 + \sigma^3 y + 3\mu^2 - \nu\sigma^2 \ ,$$

while the resultant between the first and the third equations is

$$R_2 := 3\sigma y^3 - 6\nu y^2 + \mu\sigma y + 2\mu\nu + 3\lambda\sigma \ .$$

Therefore (43) admit three common solutions if and only if R_2 divides R_1. Since the remainder of the division of R_1 by R_2 is

$$\frac{27}{\sigma^2}(4\nu^2 - \mu\sigma^2)y^2 + \frac{1}{\sigma}(\sigma^4 - 36\mu\nu - 27\lambda\sigma)y + \frac{1}{\sigma^2}(3\mu^2\sigma^2 - \nu\sigma^4 - 36\mu\nu^2 - 54\lambda\nu\sigma)$$

it turns out to be 0 if and only Λ satisfies the following conditions

$$\begin{aligned} 4\nu^2 - \mu\sigma^2 &= 0 \\ \sigma^4 - 36\mu\nu - 27\lambda\sigma &= 0 \\ 3\mu^2\sigma^2 - \nu\sigma^4 - 36\mu\nu^2 - 54\lambda\nu\sigma &= 0 \end{aligned} \tag{47}$$

Then the first equation gives

$$\mu = 4\frac{\nu^2}{\sigma^2}\ .$$

Therefore from the first two equations we get

$$\lambda = \frac{\sigma^3}{27} - \frac{16\nu^3}{3\sigma^3}\ .$$

The resultant of conditions (47) factors then as follows

$$\nu(4\nu - \sigma^2)(4\nu - \epsilon\sigma^2)(4\nu - \epsilon^2\sigma^2) = 0$$

where ϵ is a primitive cubic root of unity. All the solutions of (47) are then the following

$$\begin{aligned} \Lambda_0 &= (\frac{1}{27}\sigma^3, 0, 0, \sigma) \quad , \quad \Lambda_1 = (-\frac{5}{108}\sigma^3, \frac{1}{4}\sigma^2, -\frac{1}{4}\sigma^2, \sigma)\ , \\ \Lambda_2 &= (-\frac{5}{108}\sigma^3, \frac{\epsilon^2}{4}\sigma^2, -\frac{\epsilon}{4}\sigma^2, \sigma) \quad , \quad \Lambda_3 = (-\frac{5}{108}\sigma^3, \frac{\epsilon}{4}\sigma^2, -\frac{\epsilon^2}{4}\sigma^2, \sigma)\ . \end{aligned} \tag{48}$$

Let us first consider the second solution Λ_1. In this particular case the resultant R_2 becomes

$$R_2 = 3\sigma y^3 - \frac{3}{2}\sigma^2 y^2 + \frac{1}{4}\sigma^3 y - \frac{1}{72}\sigma^4 = 3\sigma\left(y - \frac{\sigma}{6}\right)^3\ ,$$

meaning that Λ_1 *is actually the base of a trivial deformation of (40)* since the deformed fiber associated with σ admits the unique singular point $(0, \sigma/6, 0, -\sigma/6)$ which is still a cusp of type (40). Moreover solutions Λ_2 and Λ_3 give trivial deformations too, since they can be obtained from Λ_1 by replacing

$$\begin{array}{llll} \text{either} & y \longmapsto \epsilon y\ ,\ w \longmapsto \epsilon^2 w & (\text{giving } \Lambda_2) \\ \text{or} & y \longmapsto \epsilon^2 y\ ,\ w \longmapsto \epsilon w & (\text{giving } \Lambda_3)\ . \end{array}$$

It remains to consider the first solution Λ_0. In this case the resultant R_2 becomes

$$R_2 = y^3 + \frac{\sigma^3}{27} = \left(y + \frac{\sigma}{3}\right)\left(y + \frac{\epsilon\sigma}{3}\right)\left(y + \frac{\epsilon^2\sigma}{3}\right),$$

then the deformed fiber $X_\sigma,\ \sigma \neq 0$, turns out to admit three distinct nodes given by

$$\left(0, -\frac{\sigma}{3}, 0, \frac{\sigma}{3}\right)\ ,\ \left(0, -\frac{\epsilon\sigma}{3}, 0, \frac{\epsilon^2\sigma}{3}\right)\ ,\ \left(0, -\frac{\epsilon^2\sigma}{3}, 0, \frac{\epsilon\sigma}{3}\right)\ . \tag{49}$$

Notice that the base curve $\Lambda_0 \subset T^1 \cong \mathbb{C}^4$ is actually the plain rational cubic curve $C = \{\sigma^3 - 27\lambda = \mu = \nu = 0\}$ (it admits a cusp "at infinity") meeting the hyperplane $S = \{\sigma = 0\}$ only in the origin, where they are *orthogonal* in the sense that a tangent vector to C in the origin is a multiple of $(0,0,0,1)$. The statement is then proved by thinking T^1 as the tangent space in the origin to the germ of complex space $\mathrm{Def}\, U_0$ representing the functor of 1^{st}–order deformation of the cusp (40), where $U_0 = \mathrm{Spec}\, \mathcal{O}_{F,0}$ and $F = x^2 - z^2 - y^3 + w^3$ is the defining polynomial. $\square$

Let $\widehat{X} \xrightarrow{\phi} X$ be one of the six small resolutions constructed in Proposition 3.1 and consider the *localization near to* $p \in \mathrm{Sing}(X)$

$$\begin{tikzcd} \widehat{U}_p := \phi^{-1}(U_p) \arrow[r, hook] \arrow[d, "\varphi"] & \widehat{X} \arrow[d, "\phi"] \\ U_p := \mathrm{Spec}\,\mathcal{O}_{F,p} \arrow[r, hook] & X \end{tikzcd} \tag{50}$$

which induces the following commutative diagram between Kuranishi spaces

$$\begin{tikzcd} \mathrm{Def}(\widehat{X}) \arrow[r, "\widehat{\lambda}_p"] \arrow[d, hook, "\delta"] & \mathrm{Def}(\widehat{U}_p) \arrow[d, hook, "\delta_{loc}"] \\ \mathrm{Def}(X) \arrow[r, "\lambda_p"] & \mathrm{Def}(U_p) \cong T^1 \end{tikzcd} \tag{51}$$

where the horizontal maps are the natural localization maps while the vertical maps are *injective maps* induced by the resolution ϕ (see [50] Propositions 1.8 and 1.12, [20] Proposition (11.4)).

Theorem 3.6. *The image of the map δ_{loc} in diagram (51) is the plain cubic rational curve $C \subset T^1$ defined in Proposition 3.5. In particular this means that*

(a) $\mathrm{def}(\widehat{U}_p) = 1$,
(b) $\mathrm{Im}(\lambda_p) \cap \mathrm{Im}(\delta_{loc}) = 0$,
(c) $\mathrm{Im}(\widehat{\lambda}_p) = 0$.

Proof. By the construction of the resolution $\widehat{X} \xrightarrow{\phi} X$ a deformation of U_p is induced by a deformation of $\widehat{U}_p$ if and only if it respects the factorization (38) of the local equation (31). A general deformation respecting such a factorization can be written as follows

$$(X - U + \xi)(X + U + \upsilon) = (Y - V + \alpha)(Y - \epsilon V + \beta)(Y - \epsilon^2 V + \gamma) \tag{52}$$

for $(\xi, \upsilon, \alpha, \beta, \gamma) \in \mathbb{C}^5$. After the translation $X \longmapsto X - \xi, U \longmapsto U - \upsilon$ and some elementary calculation we get the following

$$\begin{aligned} F_{\mathbf{a}} \;:=\; & F - (\alpha + \beta + \gamma)Y^2 - (\alpha + \epsilon^2\beta + \epsilon\gamma)V^2 - (\alpha + \epsilon\beta + \epsilon^2\gamma)YV \\ & -(\alpha\gamma + \alpha\beta + \beta\gamma)Y + (\beta\gamma + \epsilon\alpha\gamma + \epsilon^2\alpha\beta)V - \alpha\beta\gamma \end{aligned} \tag{53}$$

where $F := X^2 - U^2 - Y^3 + V^3$ and $\mathbf{a} := (\alpha, \beta, \gamma) \in \mathbb{C}^3$. Consider the (non–versal) deformation $\mathcal{U} \longrightarrow \mathbb{C}^3(\mathbf{a})$. The following facts then occur:

(1) *the deformed fibre $U_{\mathbf{a}} = \{F_{\mathbf{a}} = 0\}$ is isomorphic to the central fibre U_0 if and only if $\mathbf{a}$ is a point of the plane $\pi = \{\alpha + \epsilon\beta + \epsilon^2\gamma = 0\} \subset \mathbb{C}^3$; in particular $\mathcal{U}|_\pi \longrightarrow \pi$ is a trivial deformation;*
(2) *the open subset $V := \mathbb{C}^3 \setminus \pi$ is the base of a deformation of the cusp $U_0 = \{F = 0\}$ to 3 distinct nodes;*
(3) *there exists an algebraic morphism $f : \mathbb{C}^3 \longrightarrow T^1$ to the Kuranishi space $T^1 = \mathbb{C}^4(\lambda, \mu, -\nu, \sigma)$ such that $\mathrm{Im}\, f$ turns out to be precisely the plane rational cubic curve C defined in Proposition 3.5 and parameterizing the deformation of the cusp (40) to three distinct nodes.*

Let us postpone the proof of these facts to observe that fact (3) means that the deformation $\mathcal{U} \longrightarrow \mathbb{C}^3(\mathbf{a})$ is the *pull–back by* f of a versal deformation $\mathcal{V} \longrightarrow C$ i.e. $\mathcal{U} = \mathbb{C}^3 \times_C \mathcal{V}$. Then $C = \mathrm{Im}(\delta_{loc})$ proving the first part of the statement. Point (a) then follows by recalling that δ_{loc} is injective. Moreover Propositions 3.3 and 3.5 allow to conclude point (b). At last point (c) follows by (b), the injectivity of δ_{loc} and the commutativity of diagram (51).
Let us then prove facts (1), (2) and (3) stated above.

(1) , (2) : this facts are obtained analyzing the common solutions of

$$F_{\mathbf{a}} = \partial_X F_{\mathbf{a}} = \partial_Y F_{\mathbf{a}} = \partial_U F_{\mathbf{a}} = \partial_V F_{\mathbf{a}} = 0 \ .$$

Since $\partial_X F_{\mathbf{a}} = 2X$ and $\partial_U F_{\mathbf{a}} = 2U$, we can immediately reduce to look for the common solutions $(0, Y, 0, V)$ of

$$F_{\mathbf{a}}(0, Y, 0, V) = \partial_Y F_{\mathbf{a}} = \partial_V F_{\mathbf{a}} = 0 \ . \tag{54}$$

After some calculations one finds that these common solutions are given by

$$\begin{aligned} p_1 = &\quad \left(0, \tfrac{\varepsilon\beta-\gamma}{1-\epsilon}, 0, \tfrac{\beta-\gamma}{\varepsilon(1-\varepsilon)}\right) \\ p_2 = &\quad \left(0, \tfrac{\varepsilon^2\alpha-\gamma}{1-\epsilon^2}, 0, \tfrac{\alpha-\gamma}{1-\varepsilon^2}\right) \\ p_3 = &\quad \left(0, \tfrac{\varepsilon\alpha-\beta}{1-\epsilon}, 0, \tfrac{\alpha-\beta}{1-\varepsilon}\right) \end{aligned} \tag{55}$$

which have to be necessarily distinct since

$$\begin{aligned} \frac{\beta-\gamma}{\varepsilon(1-\varepsilon)} = \frac{\alpha-\gamma}{1-\varepsilon^2} \iff \frac{\alpha-\gamma}{1-\varepsilon^2} = \frac{\alpha-\beta}{1-\varepsilon} \iff \frac{\alpha-\beta}{1-\varepsilon} = \frac{\beta-\gamma}{\varepsilon(1-\varepsilon)} \\ \iff \alpha + \epsilon\beta + \epsilon^2\gamma = 0 \ . \end{aligned}$$

On the other hand, if $\alpha + \epsilon\beta + \epsilon^2\gamma = 0$ then we get the unique singular point $p_1 = p_2 = p_3$ which is still a threefold cusp.

(3) : Look at the definition (53) of $F_{\mathbf{a}}$ and construct f as a composition $f = i \circ p$ where

- $p : \mathbb{C}^3 \longrightarrow \mathbb{C}^3$ *is a linear map of rank 1 whose kernel is the plane* $\pi \subset \mathbb{C}^3$ *defined in (1),*
- $i : \mathbb{C}^3 \longrightarrow T^1$ *is the map* $(\alpha, \beta, \gamma) \in \mathbb{C}^3 \mapsto (\lambda, \mu, -\nu, \sigma) \in T^1$ *given by*

$$\begin{aligned} \lambda = -\alpha\beta\gamma \quad &, \quad \mu = -\alpha\gamma - \alpha\beta - \beta\gamma \ , \\ \nu = -\beta\gamma - \epsilon\alpha\gamma - \epsilon^2\alpha\beta \quad &, \quad \sigma = -\alpha - \epsilon\beta - \epsilon^2\gamma \ ; \end{aligned}$$

then, by (2) and Proposition 3.5, necessarily $\mathrm{Im}\, i = C$ *and* $i|_{\mathrm{Im}\, p}$ *is the rational parameterization* Λ_0 *given in (48).*

The linear map p has to annihilate the coefficients of Y^2 and V^2 in (53) i.e.

$$\alpha + \beta + \gamma = \alpha + \epsilon^2\beta + \epsilon\gamma = 0 \ .$$

Then we get the following conditions

$$\mathrm{Im}\, p = \left\langle (\epsilon, 1, \epsilon^2) \right\rangle_{\mathbb{C}} \subset \mathbb{C}^3 \ , \ \ker p = \pi = \left\langle (-\epsilon, 1, 0), (-\epsilon^2, 0, 1) \right\rangle_{\mathbb{C}} \subset \mathbb{C}^3$$

which determine p, up to a multiplicative constant $k \in \mathbb{C}$, as the linear map represented by the rank 1 matrix $k \cdot \begin{pmatrix} 1 & \epsilon & \epsilon^2 \\ \epsilon^2 & 1 & \epsilon \\ \epsilon & \epsilon^2 & 1 \end{pmatrix}$. Then

$$p(\mathbf{a}) = k(\epsilon^2\alpha + \beta + \epsilon\gamma) \cdot \left(\epsilon, 1, \epsilon^2\right)$$

and

$$f(\mathbf{a}) = i \circ p\ (\mathbf{a}) = \left(-k^3(\alpha + \epsilon\beta + \epsilon^2\gamma)^3, 0, 0, -3k(\alpha + \epsilon\beta + \epsilon^2\gamma)\right)$$

which satisfies equations $\sigma^3 - 27\lambda = \mu = \nu = 0$ of $C \subset T^1$. □

Remark 3.7. Propositions 3.3 and 3.5 and Theorem 3.6 give a detailed and revised version of what observed by Y.Namikawa in [30] Examples 1.10 and 1.11 and Remark 2.8. In fact point (c) of Theorem 3.6 means that *any global deformation of the small resolution $\widehat{X}$ induces only trivial local deformations of a neighborhood of the exceptional fibre $\phi^{-1}(p)$ over a cusp $p \in \mathrm{Sing}(X)$.*

3.2. **Picard and Kuranishi numbers.** Let $\mathcal{W} \xrightarrow{w} B$ be the universal family of bi–cubic hypersurfaces in $\mathbb{P}^2 \times \mathbb{P}^2$ (it exists by Theorem 1.4). A general choice of $b \in B$ corresponds, up to isomorphism, with a general choice of a defining polynomial in $H^0(\mathcal{O}_{\mathbb{P}^2\times\mathbb{P}^2}(3,3))$ for $W_b := w^{-1}(b)$, which turns out to be smooth. Let then $D \subset B$ be the closed subset parameterizing bi–cubic hypersurfaces defined as in (12). For a general choice of $t \in D$, the bi-cubic $W_t := w^{-1}(t)$ corresponds with a general choice of $a, b, a', b' \in H^0(\mathcal{O}_{\mathbb{P}^2}(3))$ in the defining equation (12): then $\mathrm{Sing}(W_t) = \{81 \text{ nodes}\}$. Let $X_t \xrightarrow{\psi_t} W_t$ be the small resolution of W_t obtained by taking X_t as the strict transform of W_t in the blow–up $\mathbb{P}_t$ of $\mathbb{P}^2 \times \mathbb{P}^2$ along the 9 planes $a = b = 0$, as in (13). When t varies in D we get a family of morphisms

$$\begin{tikzcd} \mathcal{X} \arrow[rr, "\Psi"] \arrow[dr, "x"'] & & \mathcal{W}|_D \arrow[dl, "w"] \\ & D & \end{tikzcd} \tag{56}$$

such that, for a general choice of $t \in D$, the small resolution $X_t := x^{-1}(t)$ is a smooth Calabi–Yau threefold.

At last let $K \subset D$ be the closed subset parameterizing bi–cubic hypersurfaces defined by taking $a' = a$ and $b' = b$ in (12). For a general choice of $k \in K$, the bi-cubic $W_k := w^{-1}(k)$ corresponds with a general choice of $a, b \in H^0(\mathcal{O}_{\mathbb{P}^2}(3))$ in the defining equation

$$\mathbb{P}^2[x] \times \mathbb{P}^2[x'] \supset W_k \ : \ a(x)b(x') - a(x')b(x) = 0\ . \tag{57}$$

Then $|\,\mathrm{Sing}(W_k)| = 81 + \nu$ with $0 \leq \nu \leq 12$ and, for k sufficiently general, $\mathrm{Sing}(W_k) = \{93 \text{ nodes}\}$. Moreover $X_k \xrightarrow{\psi_k} W_k$ is a partial small resolution such that $|\,\mathrm{Sing}(X_k)| = \nu$ and, for $k \in K$ sufficiently general, $\mathrm{Sing}(X_k) = \{12 \text{ nodes}\}$. In particular X_k turns out to be the fiber product of a rational elliptic surface with itself i.e. $X_k = Y_k \times_{\mathbb{P}^1} Y_k$. Let $Z_k \xrightarrow{\varphi_k} X_k$ be the small resolution obtained by taking Z_k as the strict transform of X_k in the blow–up of $\mathbb{P}(\mathcal{E}_k) \times \mathbb{P}(\mathcal{E}_k) \times \mathbb{P}^1$ along the diagonal locus Δ_k, where $\mathcal{E}_k$ is the bundle with respect to Y_k can be expressed

in Weierstrass form, as in Theorem 2.8. When k varies in K we get then a family of morphisms

$$
\begin{tikzcd}
\mathcal{Z} \arrow[r, "\Phi"] \arrow[dr, "z"'] & \mathcal{X}|_K \arrow[r, "\Psi"] \arrow[d, "x"] & \mathcal{W}|_K \arrow[dl, "w"] \\
& K &
\end{tikzcd}
\tag{58}
$$

such that, for a general choice of $k \in K$, the small resolution $Z_k := z^{-1}(k)$ is a smooth Calabi–Yau threefold.

Since the Namikawa fiber product X is isomorphic to the small (partial) resolution, defined as in (14), of a bi–cubic $W \subset \mathbb{P}^2 \times \mathbb{P}^2$, we can assume that there exists $0 \in K \subset D \subset B$ such that $X \cong x^{-1}(0)$. Moreover X is expressed in Weierstrass form by the bundles homomorphism $(0, B)$ given in (34), hence its singular locus $\mathrm{Sing}(X)$ is composed by $\nu = 6$ threefold cusps of local type (40). Define $\widetilde{Z} := Z_k$, $\widetilde{X} := X_t$ and $\widetilde{W} := W_b$ for k, t, b sufficiently general in K, D, B, respectively. Then $\widetilde{Z}, \widetilde{X}, \widetilde{W}$ are smooth Calabi–Yau threefolds and we get the following composition of *conifold transitions*

$$
\begin{tikzcd}
\widehat{X} = \widehat{X}_{i=0} \arrow[d, "\gamma"] \arrow[dr, dotted, "\text{conifold transition}", "T_1"'] & & & \\
Z = X_{i=0} \arrow[r, leftrightsquigarrow, "\text{smoothing}"'] \arrow[d, "\varphi=\varphi_0"] & \widetilde{Z} \arrow[d, "\widetilde{\varphi}=\varphi_k"] \arrow[dr, dotted, "\text{conifold transition}", "T_2"'] & & \\
X \arrow[r, leftrightsquigarrow, "\text{smoothing}"'] \arrow[d, "\psi=\psi_0"] & X_k \arrow[r, leftrightsquigarrow, "\text{smoothing}"'] \arrow[d, "\psi_k"] & \widetilde{X} \arrow[d, "\widetilde{\psi}=\psi_t"] \arrow[dr, dotted, "\text{conifold transition}", "T_3"'] & \\
W_0 \arrow[r, leftrightsquigarrow, "\text{smoothing}"'] & W_k \arrow[r, leftrightsquigarrow, "\text{smoothing}"'] & W = W_t \arrow[r, leftrightsquigarrow, "\text{smoothing}"'] & \widetilde{W}
\end{tikzcd}
\tag{59}
$$

where $\widehat{X} \xrightarrow{\gamma} Z \xrightarrow{\varphi} X$ is the small resolution of the cuspidal fiber product X corresponding with the choice $i = 0$ in Proposition 3.1. Precisely φ and γ are induced by the blow–ups of Δ and $(\tau\Delta)_0$, respectively. Observe that the T_2 and T_3 are precisely the conifold transitions studied in Remark 2.2 (a) and (b), respectively.

Theorem 3.8. *Assume that b, t, k are generic points in B, D, K, respectively, and that $0 \in K \subset D \subset B$ is the special point such that $X = x^{-1}(0)$ is a Namikawa cuspidal fiber product. Then the following table summarizes the numbers associated*

with the vertexes of diagram (59):

(60)

Variety	def	b_3	ρ	b_4	*defect*	χ
$\widehat{X}$	*3*	*8*	*21*	*21*	*0*	*36*
Z	*8*	*13*	*20*	*21*	*1*	*30*
$\widetilde{Z} = Z_k$	*8*	*18*	*20*	*20*	*0*	*24*
X	*19*	*18*	*19*	*21*	*2*	*24*
X_k	*19*	*29*	*19*	*20*	*1*	*12*
$\widetilde{X} = X_t$	*19*	*40*	*19*	*19*	*0*	*0*
W_0	*83*	*82*	*2*	*21*	*19*	*-57*
W_k	*83*	*93*	*2*	*20*	*18*	*-69*
W_t	*83*	*104*	*2*	*19*	*17*	*-81*
$\widetilde{W} = W_b$	*83*	*168*	*2*	*2*	*0*	*-162*

where def *is the Kuranishi number, ρ is the Picard number, b_i is the i–th Betti number, χ is the Euler–Poincaré characteristic and the* defect *is defined in Remark 1.14 (b).*

Proof. Lemma 2.4 gives the last row in table (60). The Euler–Poincaré characteristic is then easily computed by point (4) in Proposition 1.9 and point (5) in Proposition 1.12. In fact the last column on the right in table (60) follows by Proposition 2.3, and in particular by equations (16) and (15) with $\nu = 12$, with the exception of $W_k, W_0, X, Z, \widehat{X}$. To compute $\chi(W_k)$, for general $k \in K$, consider the conifold transition $T(\widetilde{Z}, W_k, \widetilde{W})$, obtained by composing T_2 and T_3. Proposition 1.9(4) gives then $\chi(W_k) = \chi(\widetilde{W}) + 93 = -69$, since $\mathrm{Sing}(W_k) = \{93 \text{ nodes}\}$. Similarly the conifold transition $T(\widehat{X}, Z, \widetilde{Z})$ gives $\chi(\widehat{X}) = \chi(Z) + 6 = \chi(\widetilde{Z}) + 12 = 36$, since $\mathrm{Sing}(Z) = \{6 \text{ nodes}\}$. To compute $\chi(X)$ we have now to apply Proposition 1.12(5) to the composition of T_1 and T_2 which is the small g.t. $T(\widehat{X}, X, \widetilde{X})$. In fact $\mathrm{Sing}(X) = \{6 \text{ cusps}\}$ and the Milnor number of a threefold cusp (40) is 4. Then $\chi(X) = \chi(\widetilde{X}) + 6 \cdot 4 = 24$. At last $\chi(W_0)$ follows by considering the small g.t. $T(\widehat{X}, W_0, \widetilde{W})$ obtained by composing T_1, T_2 and T_3. Now $\mathrm{Sing}(W_0) = \{81 \text{ nodes}\} \cup \{6 \text{ cusps}\}$ then Proposition 1.12(5) gives $\chi(W_0) = \chi(\widetilde{W}) + 81 + 6 \cdot 4 = -57$.
By definition the Kuranishi number is the maximum dimension of the Kuranishi space parameterizing the versal family. Since $\mathcal{W} \xrightarrow{w} B$ is a universal family and the Kuranishi spaces $K \subset D \subset B$ are all smooth, by the Namikawa extension of the Bogomolov–Tian–Todorov–Ran Theorem ([29] Theorem A), then

$$\begin{aligned}
\operatorname{def}(W_0) &= \operatorname{def}(W_k) = \operatorname{def}(W_t) = \operatorname{def}(\widetilde{W}) = h^{2,1}(\widetilde{W}) = 83 = \dim B \\
\operatorname{def}(X) &= \operatorname{def}(X_k) = \operatorname{def}(\widetilde{X}) = h^{2,1}(\widetilde{X}) = \dim D \qquad (61)\\
\operatorname{def}(Z) &= \operatorname{def}(\widetilde{Z}) = h^{2,1}(\widetilde{Z}) = \dim K\ .
\end{aligned}$$

On the other hand by (8) the Picard number ρ is invariant by families too, which is

$$\begin{aligned} \rho(W_0) &= \rho(W_k) = \rho(W_t) = \rho(\widetilde{W}) = h^{1,1}(\widetilde{W}) = 2 \\ \rho(X) &= \widetilde{X_k} = \rho(\widetilde{X}) = h^{1,1}(\widetilde{X}) \\ \rho(Z) &= \rho(\widetilde{Z}) = h^{1,1}(\widetilde{Z}) \\ \rho(\widehat{X}) &= h^{1,1}(\widehat{X}) \ . \end{aligned} \tag{62}$$

Let us first of all determine $\operatorname{def}(X)$ by computing the moduli of X. Recall that $X = Y \times_{\mathbb{P}^1} Y$ with Y rational elliptic surface with section and six cuspidal fibers. The moduli of Y are 8 since they are given by the moduli of an elliptic pencil in $\mathbb{P}^2$. Moreover the six cuspidal fibers are parameterized by the roots in $\mathbb{P}^1$ of a general element in $H^0(\mathcal{O}_{\mathbb{P}^1}(6))$ up to the action of the projective group $\mathbb{P}\,\mathrm{GL}(2)$. Then

$$\operatorname{def}(X) = 2 \cdot \operatorname{def}(Y) + h^0(\mathcal{O}_{\mathbb{P}^1}(6)) - \dim \mathrm{GL}(2,\mathbb{C}) = 2\cdot 8 + 7 - 4 = 19 \ .$$

Consider the conifold transition $T_3 = T(\widetilde{X}, W_t, \widetilde{W})$ and recall points (1) and (3) of the Proposition 1.9. Since $|\operatorname{Sing}(W_t)| = 81$ and $h^{2,1}(\widetilde{W}) - h^{2,1}(\widetilde{X}) = 83 - 19 = 64$ then the integers associated with T_3 are $c = 64$ and $k = 81 - 64 = 17$. Then, by point (2) of Proposition 1.9, we are able to compute all the numbers pertinent to W_t and $\widetilde{X}$ in table (60).

To apply the same argument to the conifold transitions T_2 and T_1 in diagram (59) let us determine the defect of X_k and of Z which actually turns out to be the relative Picard numbers $\rho(\widetilde{Z}/X_k)$ and $\rho(\widehat{X}/Z)$, respectively. The resolution $\widetilde{\varphi} : \widetilde{Z} \to X_k$ is obtained by blowing up the diagonal locus Δ as explained in the proof of Proposition 2.1. Then $\operatorname{Pic}(\widetilde{Z})$ is generated by the pull–back $\widetilde{\varphi}^*(\operatorname{Pic}(X_k))$ and by the strict transform of the divisor $\Delta \cap X_k$. Therefore $\rho(\widetilde{Z}/X_k) = 1$. The same argument applies to the resolution $\gamma : \widehat{X} \to Z$ since it is obtained as the blow–up of $(\tau\Delta)_0$, as explained in the proof of Proposition 3.1. Then also $\rho(\widehat{X}/Z) = 1$. This is enough to conclude that the integers associated with T_2 are $k = 1, c = |\operatorname{Sing}(X_k)| - 1 = 11$ while those associated with T_1 are $k = 1, c = |\operatorname{Sing}(Z)| - 1 = 5$. Proposition 1.9(2) allows then to compute all the numbers in table (60) pertinent to $X_k, \widetilde{Z}, Z, \widehat{X}$.

For what concerns W_k consider once again the conifold transition $T(\widetilde{Z}, W_k, \widetilde{W})$: we are now able to determine the associated integers as the sum of those associated with T_2 and T_3. Precisely $k = 1+17 = 18$ and $c = |\operatorname{Sing}(W_k)| - 18 = 75 (= 64+11)$ and the numbers in table (60) pertinent to W_k follows once again by Proposition 1.9(2).

At last the computation of numbers pertinent to X and W_0 needs to employ Proposition 1.12. Precisely let us consider the already mentioned small g.t.s $T(\widehat{X}, X, \widetilde{X})$ and $T(\widehat{X}, W_0, \widetilde{W})$. The integers associated with the first one turns out to be $k = 1+1 = 2, c' = 12-2 = 10, c'' = 24-2 = 22$. Those associated with the second one are $k = 1+1+17 = 19, c' = 93-19 = 74, c'' = 105-19 = 86$. Proposition 1.12(3) ends up the proof. $\square$

Remark 3.9. As observed in proving Theorem 3.8 the relative Picard number of the resolution $\widetilde{\psi} : \widetilde{X} \to W_t$ is $\rho(\widetilde{X}/W_t) = 17$. Let us give here a quick geometric explanation of this number.

First of all $\widetilde{\psi}$ is obtained as the blow–up of nine $\mathbb{P}^2$'s in $\mathbb{P}^2 \times \mathbb{P}^2$ parameterized by *the base locus of an elliptic pencil* $a(x) = b(x) = 0$. Then only 8 of the 9

exceptional divisors are to be considered independent. Moreover the 81 exceptional $\mathbb{P}^1$'s comes arranged nine by nine in the 9 exceptional divisors. This means that we get $9 - 1 = 8$ independent conditions for each independent exceptional divisor i.e. 64 independent conditions on the 81 exceptional $\mathbb{P}^1$'s. Then precisely $81 - 64 = 17$ of them turns out to be independent.

Remark 3.10. Observe that, by construction, *the general fiber* $\widetilde{X}$ *of the family* $\mathcal{X} \longrightarrow D$ *is a fiber product* $\widetilde{X} = Y \times_{\mathbb{P}^1} Y'$ *of two general rational elliptic surfaces with sections.* Then, in particular, Theorem 3.8 gives an alternative argument to prove that $\mathrm{Pic}(\widetilde{X})$ has rank 19, which is a known fact (see [9] Corollary 3.2).

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Dipartimento di Matematica, Università di Torino, via Carlo Alberto 10, 10123 Torino

E-mail address: `michele.rossi@unito.it`